CS 4110

Programming Languages & Logics

Lecture 19
Continuations

Continuations

In the preceding translations, the control structure of the source language was translated directly into the corresponding control structure in the target language.

For example:

$$\mathcal{T}[\![\lambda x.\, e]\!] = \lambda x.\, \mathcal{T}[\![e]\!]$$
$$\mathcal{T}[\![e_1\ e_2]\!] = \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!]$$

What can go wrong with this approach?

Continuations

- A snippet of code that represents “the rest of the program”
- Can be used directly by programmers...
- ...or in program transformations by a compiler
- Make the control flow of the program explicit
- Also useful for defining the meaning of features like exceptions

Example

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$$k_1 = \lambda a.\, k_0\, (a - 4)$$

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k_0 &= \lambda v.\, (\lambda x.\, x)\, v \\
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k_2 &= \lambda b.\, k_1\, (3 * b)
\end{aligned}$$

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The original expression is equivalent to $k_3\ 1$, or:

$$(\lambda c.\, (\lambda b.\, (\lambda a.\, (\lambda v.\, (\lambda x.\, x)\, v)\, (a - 4))\, (3 * b))\, (c + 2))\, 1$$

Example (Continued)

Recall that let $x = e$ in e' is syntactic sugar for $(\lambda x.\, e')\; e$.

Hence, we can rewrite the expression with continuations more succinctly as

$$
\begin{array}{l}
\text{let } c = 1 \text{ in} \\
\text{let } b = c + 2 \text{ in} \\
\text{let } a = 3 * b \text{ in} \\
\text{let } v = a - 4 \text{ in} \\
(\lambda x.\, x)\; v
\end{array}
$$

CPS Transformation

We write $\mathcal{CPS}[\![e]\!]\, k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

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\mathcal{CPS}[\![e_1 + e_2]\!]\, k &= \mathcal{CPS}[\![e_1]\!]\, (\lambda n.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda m.\, k\, (n + m)))
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\mathcal{CPS}[\![\lambda x.\,e]\!]\,k &= k\,(\lambda x.\,\lambda k'.\,\mathcal{CPS}[\![e]\!]\,k')
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\mathcal{CPS}[\![e_1\ e_2]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda f.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda v.\,f\,v\,k))
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We can also translate other language features, like products:

$$e ::= \cdots \mid (e_1, e_2) \mid \#1\, e \mid \#2\, e$$

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\mathcal{CPS}[\![\#1\ e]\!]\, k &= \mathcal{CPS}[\![e]\!]\, (\lambda v.\, k\, (\#1\ v)) \\
\mathcal{CPS}[\![\#2\ e]\!]\, k &= \mathcal{CPS}[\![e]\!]\, (\lambda v.\, k\, (\#2\ v))
\end{aligned}$$