Lecture 8
Denotational Semantics Proofs
Kleene Fixed-Point Theorem

Definition (Scott Continuity)

A function $F$ is \textit{Scott-continuous} if for every chain $X_1 \subseteq X_2 \subseteq \ldots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$. 
Kleene Fixed-Point Theorem

Definition (Scott Continuity)

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Theorem (Kleene Fixed Point)

Let $F$ be a Scott-continuous function. The least fixed point of $F$ is $\bigcup_i F^i(\emptyset)$. 
\[C[\text{skip}] = \{(\sigma, \sigma)\}\]
\[C[x := a] = \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in A[a]\}\]
\[C[c_1; c_2] = \{(\sigma, \sigma') \mid \exists \sigma''. ((\sigma, \sigma'') \in C[c_1] \land (\sigma'', \sigma') \in C[c_2])\}\]
\[C[\text{if } b \text{ then } c_1 \text{ else } c_2] = \{(\sigma, \sigma') \mid (\sigma, \text{true}) \in B[b] \land (\sigma, \sigma') \in C[c_1]\} \cup \{(\sigma, \sigma') \mid (\sigma, \text{false}) \in B[b] \land (\sigma, \sigma') \in C[c_2]\}\]
\[C[\text{while } b \text{ do } c] = \text{fix}(f)\]
where \(F(f) = \{(\sigma, \sigma) \mid (\sigma, \text{false}) \in B[b]\} \cup \{(\sigma, \sigma') \mid (\sigma, \text{true}) \in B[b] \land \exists \sigma''. ((\sigma, \sigma'') \in C[c] \land (\sigma'', \sigma') \in f)\}\]
Exercises

\texttt{skip}; c \text{ and } c; \texttt{skip} are equivalent.
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$C[\texttt{while false do } c]$ is equivalent to...
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$C[\text{while false do } c]$ is equivalent to skip.
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skip; c and c; skip are equivalent.

C[while false do c] is equivalent to skip.

C[while true do skip] = ?