Lecture 5
IMP Properties
Intuitively, two commands are equivalent if they produce the same result under any store...

**Definition (Equivalence of commands)**

Two commands $c$ and $c'$ are equivalent (written $c \sim c'$) if, for any stores $\sigma$ and $\sigma'$, we have

$$\langle \sigma, c \rangle \downarrow \sigma' \iff \langle \sigma, c' \rangle \downarrow \sigma'.$$
Command Equivalence

For example, we can prove that every \texttt{while} command is equivalent to its “unrolling”:

**Theorem**

For all $b \in \text{Bexp}$ and $c \in \text{Com}$,

\[
\text{while } b \text{ do } c \sim \text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}
\]

**Proof.**

We show each implication separately...
IMP Questions

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  A: `while true do skip`

- Q: Doesthis mean that IMP is Turing complete?
  
  A: Not quite... we also need to check the language is not finite state... but IMP has real mathematical integers.

- Q: What if we replace `Int` with `Int64`?
  
  A: Then we would lose Turing completeness.

- Q: How much space do we need to represent configurations during execution of an IMP program?
  
  A: Can calculate a fixed bound!
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Determinism

**Theorem**

\[ \forall c \in \text{Com}, \sigma, \sigma', \sigma'' \in \text{Store}. \]

*if* \( \langle \sigma, c \rangle \downarrow \sigma' \) and \( \langle \sigma, c \rangle \downarrow \sigma'' \) *then* \( \sigma' = \sigma'' \).
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\[ \forall c \in \text{Com}, \sigma, \sigma', \sigma'' \in \text{Store}. \]
\[ \text{if } \langle \sigma, c \rangle \Downarrow \sigma' \text{ and } \langle \sigma, c \rangle \Downarrow \sigma'' \text{ then } \sigma' = \sigma''. \]

Proof.
By structural induction on \( c \)...
Determinism

**Theorem**

\[ \forall c \in \text{Com}, \sigma, \sigma', \sigma'' \in \text{Store}. \]

If \( \left\langle \sigma, c \right\rangle \downarrow \sigma' \) and \( \left\langle \sigma, c \right\rangle \downarrow \sigma'' \) then \( \sigma' = \sigma'' \).

**Proof.**

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**Proof.**

By induction on the derivation of \( \left\langle \sigma, c \right\rangle \downarrow \sigma' \)...
Write $\mathcal{D} \models y$ if the conclusion of derivation $\mathcal{D}$ is $y$.  
(Read as “$\mathcal{D}$ proves $y$.”)
Derivations

Write $\mathcal{D} \vdash y$ if the conclusion of derivation $\mathcal{D}$ is $y$. (Read as “$\mathcal{D}$ proves $y$.”)

Example:

Given the derivation,

\[
\begin{array}{c}
\langle \sigma, 6 \rangle \downarrow 6 \\
\langle \sigma, 7 \rangle \downarrow 7
\end{array}
\]

\[
\frac{\langle \sigma, 6 \times 7 \rangle \downarrow 42}{\langle \sigma, i := 6 \times 7 \rangle \downarrow \sigma[i \mapsto 42]}
\]

we would write: $\mathcal{D} \vdash \langle \sigma, i := 42 \rangle \downarrow \sigma[i \mapsto 42]$
Induction on Derivations

Remember that every “true” fact given by an inductive definition must have a derivation that “proves” that fact.

For many inductive proofs, it’s useful to visualize the derivation tree for each fact.
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For many inductive proofs, it’s useful to visualize the derivation tree for each fact.

In each case in an inductive proof, we assume that the property $P$ holds for the rule’s premises and prove it for the rule’s conclusion.

Those premises each also have derivations.

A derivation $D'$ is an immediate subderivation of $D$ if $D' \vdash z$ where $z$ is one of the premises used of the final rule of derivation $D$. 
Large-Step Semantics

<table>
<thead>
<tr>
<th>Rule</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
<td><strong>Skip</strong></td>
<td>[\langle \sigma, \text{skip} \rangle \downarrow \sigma]</td>
</tr>
<tr>
<td><strong>ASSGN</strong></td>
<td>[\langle \sigma, a \rangle \downarrow n]</td>
</tr>
<tr>
<td><strong>SEQ</strong></td>
<td>[\langle \sigma, c_1 \rangle \downarrow \sigma'] , [\langle \sigma', c_2 \rangle \downarrow \sigma'']</td>
</tr>
<tr>
<td><strong>IF-T</strong></td>
<td>[\langle \sigma, \text{if } b \text{ then } c_1 \text{ else } c_2 \rangle \downarrow \sigma']</td>
</tr>
<tr>
<td><strong>IF-F</strong></td>
<td>[\langle \sigma, \text{if } b \text{ then } c_1 \text{ else } c_2 \rangle \downarrow \sigma']</td>
</tr>
<tr>
<td><strong>WHILE-T</strong></td>
<td>[\langle \sigma, b \rangle \downarrow \text{true}] , [\langle \sigma, c \rangle \downarrow \sigma'] , [\langle \sigma', \text{while } b \text{ do } c \rangle \downarrow \sigma'']</td>
</tr>
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<td><strong>WHILE-F</strong></td>
<td>[\langle \sigma, \text{while } b \text{ do } c \rangle \downarrow \sigma]</td>
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