Lecture 2
Introduction to Semantics
Question: What is the meaning of a program?
Semantics

**Question:** What is the meaning of a program?

**Answer:** We could execute the program using an interpreter or a compiler, or we could consult a manual...

...but none of these is a satisfactory solution.
Formal Semantics

Three Approaches

- **Operational**
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters
  - \( \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \)

- **Denotational:**
  - Model program as mathematical objects
  - Useful for theoretical foundations
  - \([e]\)

- **Axiomatic**
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness
  - \( \vdash \{ \phi \} e \{ \psi \} \)
Arithmetic Expressions
Syntax

A language of integer arithmetic expressions with assignment.
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Metavariables:

\[ x, y, z \in \text{Var} \]
\[ n, m \in \text{Int} \]
\[ e \in \text{Exp} \]
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BNF Grammar:

\[
e ::= x \\
    | n \\
    | e_1 + e_2 \\
    | e_1 * e_2 \\
    | x := e_1 ; e_2
\]
What expression does the string “1 + 2 * 3” describe?
Ambiguity

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There are two possible parse trees:

```
+       *
  |     |
  1     2
  |     |
  *     3

+       +
  |     |
  1     2
  |     |
  3
```
Ambiguity

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In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).
Representing Expressions

BNF Grammar:

\[ e ::= x \]
\[ \mid n \]
\[ \mid e_1 + e_2 \]
\[ \mid e_1 \times e_2 \]
\[ \mid x := e_1 ; e_2 \]
Representing Expressions

BNF Grammar:

\[ e ::= x \]
\[ n \]
\[ e_1 + e_2 \]
\[ e_1 * e_2 \]
\[ x := e_1 ; e_2 \]

OCaml:

```ocaml
type exp = Var of string |
| Int of int |
| Add of exp * exp |
| Mul of exp * exp |
| Assgn of string * exp * exp |
```

Example: Mul(Int 2, Add(Var ”foo”, Int 1))
Representing Expressions

BNF Grammar:

\[ e ::= x \]
\[ \quad | \quad n \]
\[ \quad | \quad e_1 + e_2 \]
\[ \quad | \quad e_1 * e_2 \]
\[ \quad | \quad x ::= e_1 ; e_2 \]

Java:

abstract class Expr {}
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assign extends Expr { String var, Expr exp1, exp2; ... }

Example:  new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))
Quiz

- 7 + (4 * 2) evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 ; 2 \times 3 \times i$ evaluates to ...?
Quiz

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• $7 + (4 \times 2)$ evaluates to 15
• $i := 6 + 1; 2 \times 3 \times i$ evaluates to 42
• $x + 1$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?
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- $x + 1$ evaluates to error?

The rest of this lecture will make these intuitions precise...
Mathematical Preliminaries
Binary Relations

The *product* of two sets $A$ and $B$, written $A \times B$, contains all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. 
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Some Important Relations

- empty: $\emptyset$
- total: $A \times B$
- identity on $A$: $\{(a, a) \mid a \in A\}$.
- composition $R; S$: $\{(a, c) \mid \exists b. (a, b) \in R \land (b, c) \in S\}$
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The image of $f$ is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. Formally:

$$\text{image}(f) \triangleq \{ f(a) \mid a \in A \}$$
Some Important Functions

Given two functions $f : A \to B$ and $g : B \to C$, the composition of $f$ and $g$ is defined by: $(g \circ f)(x) = g(f(x))$  

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A partial function $f : A \to B$ is a total function $f : A' \to B$ on a set $A' \subseteq A$. The notation $\text{dom}(f)$ refers to $A'$. 
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A function $f : A \rightarrow B$ is said to be \textit{surjective} (or \textit{onto}) if and only if the image of $f$ is $B$. 
Operational Semantics
An **operational semantics** describes how a program executes on some abstract (imaginary) machine.
Overview

An operational semantics describes how a program executes on some abstract (imaginary) machine. A small-step operational semantics describes how such an execution proceeds from configuration to configuration: 

\[ \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \]
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A small-step operational semantics describes how such an execution proceeds from configuration to configuration:

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For our language, a configuration \( \langle \sigma, e \rangle \) is a pair of:

- a store \( \sigma \) that records the values of variables,
- and the expression \( e \) being evaluated.
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- a store \( \sigma \) that records the values of variables,
- and the expression \( e \) being evaluated.

More formally:

\[
\text{Store} \triangleq \text{Var} \rightarrow \text{Int} \\
\text{Config} \triangleq \text{Store} \times \text{Exp}
\]

(A store is a partial function from variables to integers.)
Operational Semantics

The small-step operational semantics itself is a relation on configurations—i.e., a subset of $\text{Config} \times \text{Config}$. 
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Notation: $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$
which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in \rightarrow$.
Operational Semantics

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**Question:** How should we define this relation?
Operational Semantics

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**Notation:** $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$

which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in \rightarrow$.

**Question:** How should we define this relation? Remember that there are an infinite number of configurations and possible steps!
Inference Rules

**Answer:** Define it inductively, using **inference rules**:

\[ \text{premise}_1 \quad \text{premise}_2 \quad \ldots \quad \begin{array}{c} \text{conclusion} \\ \hline \end{array} \quad \text{NAME} \]
Answer: Define it inductively, using inference rules:

```
premise_1 \quad \text{premise}_2 \quad \ldots
\hline
\text{conclusion} \quad \text{name}
```

An inference rule defines an implication: if all the premises hold, then the conclusion also holds.

Formally, “→” is the smallest relation that is closed under all the inference rules.
Variables

\[ n = \sigma(x) \]

\[ \langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle \] \hspace{1cm} \text{VAR}
Addition

\[ p = m + n \]

\[
\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle
\]
Addition

\[ p = m + n \]

\[ \langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle \quad \text{ADD} \]

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle \]

\[ \langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e_1' + e_2 \rangle \quad \text{LADD} \]
Addition

\[ p = m + n \]
\[ \langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle \] \text{ ADD}

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \] \text{ LADD}
\[ \langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle \]

\[ \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \] \text{ RADD}
\[ \langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e'_2 \rangle \]
Multiplication

\[ p = m \times n \]

\[ \langle \sigma, m \times n \rangle \rightarrow \langle \sigma, p \rangle \]

\text{MUL}
Multiplication

\[ p = m \times n \]
\[
\langle \sigma, m \times n \rangle \rightarrow \langle \sigma, p \rangle \quad \text{MUL}
\]

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle \quad \text{LMUL}
\]
\[
\langle \sigma, e_1 \times e_2 \rangle \rightarrow \langle \sigma', e_1' \times e_2 \rangle
\]

\[
\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle \quad \text{RMUL}
\]
\[
\langle \sigma, n \times e_2 \rangle \rightarrow \langle \sigma', n \times e_2' \rangle
\]
\[
\sigma' = \sigma[x \mapsto n] \\
\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle \quad \text{ASSGN}
\]

\textbf{Notation:} \(\sigma[x \mapsto n]\) is a \textit{new} (partial) function that mostly behaves like \(\sigma\), except that it maps \(x\) to \(n\).
Assignment

\[ \sigma' = \sigma[x \mapsto n] \]
\[ \langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle \] \hspace{1cm} \text{ASSGN}

**Notation:** \( \sigma[x \mapsto n] \) is a new (partial) function that mostly behaves like \( \sigma \), except that it maps \( x \) to \( n \).

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \]
\[ \langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle \] \hspace{1cm} \text{ASSGN1}
Operational Semantics

\[
\begin{align*}
  n = \sigma(x) & \quad \text{VAR} \\
  \langle \sigma, x \rangle & \rightarrow \langle \sigma, n \rangle \\
  \langle \sigma, e_2 \rangle & \rightarrow \langle \sigma', e'_2 \rangle \quad \text{RADD} \\
  \langle \sigma, n + e_2 \rangle & \rightarrow \langle \sigma', n + e'_2 \rangle \\
  \langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle \quad \text{LMUL} \\
  \langle \sigma, e_1 * e_2 \rangle & \rightarrow \langle \sigma', e'_1 * e_2 \rangle \\
  \langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle \quad \text{MUL} \\
  \langle \sigma, m * e_2 \rangle & \rightarrow \langle \sigma', m * e'_2 \rangle \\
  \langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle \quad \text{ASSGN1} \\
  \langle \sigma, x := e_1 ; e_2 \rangle & \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle \\
  \sigma' = \sigma[x \mapsto n] & \quad \text{ASSGN} \\
  \langle \sigma, x := n ; e_2 \rangle & \rightarrow \langle \sigma', e_2 \rangle
\end{align*}
\]
Multi-Step Evaluation

We can define the multi-step evaluation relation, written $\rightarrow^*$, as the reflexive and transitive closure of the small-step evaluation relation.

\[
\begin{align*}
\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle & \quad \text{REFL} \\
\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle & \quad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle & \quad \langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle \quad \text{TRANS}
\end{align*}
\]