Lecture 20
Normalization
Type “Completeness”? 

Are all well-behaved programs well-typed?
Normalization

The simply-typed lambda calculus enjoys a remarkable property:

Every well-typed program terminates.
Simply-Typed Lambda Calculus

Syntax

expressions  \[ e ::= x \mid \lambda x : \tau . e \mid e_1 e_2 \mid () \]
values   \[ \nu ::= \lambda x : \tau . e \mid () \]

types \[ \tau ::= \text{unit} \mid \tau_1 \to \tau_2 \]
Simply-Typed Lambda Calculus

Syntax

expressions

\[ e ::= x | \lambda x : \tau. \ e | e_1 \ e_2 | () \]

values

\[ v ::= \lambda x : \tau. \ e | () \]

types

\[ \tau ::= \text{unit} | \tau_1 \rightarrow \tau_2 \]

Dynamic Semantics

\[ E ::= [] | E \ e | v \ E \]

\[ e \rightarrow e' \]

\[ E[e] \rightarrow E[e'] \]

\[ (\lambda x : \tau. \ e) \ v \rightarrow e\{v/x\} \]
Simply-Typed Lambda Calculus

Static Semantics

\[ \Gamma \vdash () : \text{unit} \quad \text{T-UNIT} \]

\[ \Gamma(x) = \tau \quad \frac{}{\Gamma \vdash x : \tau} \quad \text{T-VAR} \]

\[ \Gamma, x : \tau \vdash e : \tau' \quad \frac{}{\Gamma \vdash \lambda x : \tau. e : \tau \to \tau'} \quad \text{T-Abs} \]

\[ \Gamma \vdash e_1 : \tau \to \tau' \quad \Gamma \vdash e_2 : \tau \quad \frac{\Gamma \vdash e_1 e_2 : \tau'}{\text{T-App}} \]
Lemma (Inversion)

- If $\Gamma \vdash x : \tau$ then $\Gamma(x) = \tau$
- If $\Gamma \vdash \lambda x : \tau_1. e : \tau$ then $\tau = \tau_1 \rightarrow \tau_2$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$.
- If $\Gamma \vdash e_1 e_2 : \tau$ then $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 : \tau'$.
Supporting Lemmas

Lemma (Inversion)

• If $\Gamma \vdash x : \tau$ then $\Gamma(x) = \tau$
• If $\Gamma \vdash \lambda x : \tau_1. e : \tau$ then $\tau = \tau_1 \rightarrow \tau_2$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$.
• If $\Gamma \vdash e_1 e_2 : \tau$ then $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 : \tau'$.

Lemma (Canonical Forms)

• If $\Gamma \vdash v : \text{unit}$ then $v = ()$
• If $\Gamma \vdash v : \tau_1 \rightarrow \tau_2$ then $v = \lambda x : \tau_1. e$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$. 
First Attempt

Theorem (Normalization)

If \( \vdash e : \tau \) then there exists a value \( v \) such that \( e \rightarrow^* v \).
Logical Relations

Idea: define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
- At function types, the set contains all expressions such that the property is preserved whenever we apply the function to an argument of appropriate type that is also in the set.

\[
\lambda \alpha. \psi \qquad \lambda \beta. \varphi
\]
Logical Relations

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- At base types the set contains all expressions satisfying some property.
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In our setting, the property will concern normalization...
Logical Relation

Definition (Logical Relation)

- \( R_{\text{unit}}(e) \) iff \( e : \text{unit} \) and \( e \) halts.
- \( R_{\tau_1 \rightarrow \tau_2}(e) \) iff \( e : \tau_1 \rightarrow \tau_2 \) and \( e \) halts, and for every \( e' \) such that \( R_{\tau_1}(e') \) we have \( R_{\tau_2}(e \ e') \).
Supporting Lemmas

Lemma

If $R_{\tau}(e)$ then e halts.
Supporting Lemmas

Lemma

If $R_{\tau}(e)$ then $e$ halts.

Lemma

If $\vdash e : \tau$ and $e \rightarrow e'$ then $R_{\tau}(e)$ iff $R_{\tau}(e')$. 
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If $\vdash e : \tau$ and $e \rightarrow e'$ then $R_\tau(e)$ iff $R_\tau(e')$.

Lemma (Goal)

If $\vdash e : \tau$ then $R_\tau(e)$
Main Lemma

**Lemma (Goal – Strengthened)**

If

- $x_1 : \tau_1, \ldots, x_k : \tau_k \vdash e : \tau$,
- $v_1$ through $v_k$ are values such that $\vdash v_1 : \tau_1$ through $\vdash v_k : \tau_k$, and
- $R_{\tau_1}(v_1)$ through $R_{\tau_k}(v_k)$,

then $R_{\tau}(e\{v_1/x_1\} \ldots \{v_k/x_k\})$. 