Lecture 2
Introduction to Semantics
Question: What is the meaning of a program?
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Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...

...but none of these is a satisfactory solution.
Formal Semantics

Three Approaches

- **Operational**
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters
  \[
  \langle \sigma, e \rangle \xrightarrow{} \langle \sigma', e' \rangle
  \]

- **Denotational:**
  - Model program as mathematical objects
  - Useful for theoretical foundations
  \[ [e] \]

- **Axiomatic**
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness
  \[ \vdash \{ \phi \} e \{ \psi \} \]
Arithmetic Expressions
Syntax

A language of integer arithmetic expressions with assignment.
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Metavariables:

\[
\begin{align*}
x, y, z & \in \text{Var} \\
n, m & \in \text{Int} \\
e & \in \text{Exp}
\end{align*}
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BNF Grammar:

\[ e ::= x \]
\[ | n \]
\[ | e_1 + e_2 \]
\[ | e_1 \times e_2 \]
\[ | x := e_1 ; e_2 \]
Ambiguity

What expression does the string “1 + 2 * 3” describe?
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There are two possible parse trees:
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In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).
Representing Expressions

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\[ e ::= x \]
\[ | n \]
\[ | e_1 + e_2 \]
\[ | e_1 * e_2 \]
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Representing Expressions

BNF Grammar:

\[
e ::= x \quad | \quad n \quad | \quad e_1 + e_2 \quad | \quad e_1 * e_2 \quad | \quad x := e_1 ; e_2
\]

OCaml:

```ocaml
type exp = Var of string
  | Int of int
  | Add of exp * exp
  | Mul of exp * exp
  | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var "foo", Int 1))
Representing Expressions

BNF Grammar:

\[
e ::= x \\
    | n \\
    | e_1 + e_2 \\
    | e_1 * e_2 \\
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\]

Java:

- abstract class Expr {}
- class Var extends Expr { String name; ... }
- class Int extends Expr { int val; ... }
- class Add extends Expr { Expr exp1, exp2; ... }
- class Mul extends Expr { Expr exp1, exp2; ... }
- class Assgn extends Expr { String var, Expr exp1, exp2; ... }

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))
Quiz

- $7 + (4 \times 2)$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 ; 2 \times 3 \times i$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1; \ 2 \times 3 \times i$ evaluates to 42
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 ; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 ; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?
Quiz

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- $i := 6 + 1 ; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?

The rest of this lecture will make these intuitions precise...
Mathematical Preliminaries
Binary Relations

The *product* of two sets $A$ and $B$, written $A \times B$, contains all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. 
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Some Important Relations

- empty: $\emptyset$
- total: $A \times B$
- identity on $A$: $\{(a, a) \mid a \in A\}$.
- composition $R; S$: $\{(a, c) \mid \exists b. (a, b) \in R \land (b, c) \in S\}$
A (total) function $f$ is a binary relation $f \subseteq A \times B$ with the property that every $a \in A$ is related to exactly one $b \in B$. 
Functions

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When $f$ is a function, we usually write $f : A \rightarrow B$ instead of $f \subseteq A \times B$. 
A (total) function $f$ is a binary relation $f \subseteq A \times B$ with the property that every $a \in A$ is related to exactly one $b \in B$.

When $f$ is a function, we usually write $f: A \to B$ instead of $f \subseteq A \times B$.

The image of $f$ is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. Formally:

$$\text{image}(f) \triangleq \{ f(a) \mid a \in A \}$$
Some Important Functions

Given two functions \( f : A \to B \) and \( g : B \to C \), the composition of \( f \) and \( g \) is defined by: \( (g \circ f)(x) = g(f(x)) \)  

Note order!
Some Important Functions

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A partial function $f : A \rightarrow B$ is a total function $f : A' \rightarrow B$ on a set $A' \subseteq A$. The notation $\text{dom}(f)$ refers to $A'$. 
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A function \( f : A \rightarrow B \) is said to be injective (or one-to-one) if and only if \( a_1 \neq a_2 \) implies \( f(a_1) \neq f(a_2) \).
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A function $f : A \rightarrow B$ is said to be surjective (or onto) if and only if the image of $f$ is $B$. 
Operational Semantics
Overview

An operational semantics describes how a program executes on some abstract (imaginary) machine.
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A **small-step** semantics describes how such an execution proceeds from configuration to configuration: $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$
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A small-step semantics describes how such an execution proceeds from configuration to configuration: \( \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \)

For our language, a configuration \( \langle \sigma, e \rangle \) is a pair of:
- a store \( \sigma \) that records the values of variables,
- and the expression \( e \) being evaluated.
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- a store \( \sigma \) that records the values of variables,
- and the expression \( e \) being evaluated.

More formally:

\[
\begin{align*}
\text{Store} & \triangleq \text{Var} \rightarrow \text{Int} \\
\text{Config} & \triangleq \text{Store} \times \text{Exp}
\end{align*}
\]

(A store is a partial function from variables to integers.)
The small-step operational semantics itself is a relation on configurations—i.e., a subset of \( \text{Config} \times \text{Config} \).
Operational Semantics

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Notation: $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$
which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in \rightarrow$.
Operational Semantics

The small-step operational semantics itself is a relation on configurations—i.e., a subset of $\text{Config} \times \text{Config}$.

**Notation:** $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$

which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in "\rightarrow"$.

**Question:** How should we define this relation?
Operational Semantics

The small-step operational semantics itself is a relation on configurations—i.e., a subset of $\text{Config} \times \text{Config}$.

**Notation:** $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$
which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in \rightarrow$.

**Question:** How should we define this relation? Remember that there are an infinite number of configurations and possible steps!
Inference Rules

**Answer:** Define it inductively, using *inference rules*:

\[
\begin{array}{cccc}
\text{premise}_1 & \text{premise}_2 & \cdots & \text{Name} \\
\hline
\text{conclusion} & \\
\end{array}
\]
Inference Rules

**Answer:** Define it inductively, using **inference rules**:

\[
\begin{array}{c}
\text{premise}_1 \\
\text{premise}_2 \\
\ldots \\
\text{conclusion} \\
\end{array} \quad \text{NAME}
\]

An inference rule defines an implication: if all the **premises** hold, then the **conclusion** also holds.

Formally, “→” is the smallest relation that is closed under all the inference rules.
Variables

\[
n = \sigma(x) \quad \text{VAR}
\]

\[
\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle
\]
Addition

\[ p = m + n \]

\[ \langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle \]  ADD
Addition

\[
p = m + n
\]

\[
\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle \quad \text{ADD}
\]

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle
\]

\[
\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle \quad \text{LADD}
\]
Addition

\[ p = m + n \]

\[ \langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle \]

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \]

\[ \langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle \]

\[ \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \]

\[ \langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e'_2 \rangle \]
Multiplication

\[ p = m \times n \]

\[ \langle \sigma, m \times n \rangle \rightarrow \langle \sigma, p \rangle \]
Multiplication

\[ p = m \times n \]

\[
\langle \sigma, m \times n \rangle \rightarrow \langle \sigma, p \rangle \quad \text{MUL}
\]

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \quad \text{LMUL}
\]

\[
\langle \sigma, e_1 \times e_2 \rangle \rightarrow \langle \sigma', e'_1 \times e_2 \rangle
\]

\[
\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \quad \text{RMUL}
\]

\[
\langle \sigma, n \times e_2 \rangle \rightarrow \langle \sigma', n \times e'_2 \rangle
\]
\[
\sigma' = \sigma[x \mapsto n]
\]
\[
\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle \tag{ASSGN}
\]

**Notation:** \(\sigma[x \mapsto n]\) is a *new* function that mostly behaves like \(\sigma\), except that it maps \(x\) to \(n\).
Assignment

\[
\sigma' = \sigma[x \mapsto n]
\]

\[
\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle \quad \text{ASSGN}
\]

**Notation:** \( \sigma[x \mapsto n] \) is a *new* function that mostly behaves like \( \sigma \), except that it maps \( x \) to \( n \).

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle
\]

\[
\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle \quad \text{ASSGN1}
\]
Operational Semantics

\[
\begin{align*}
\text{VAR} & \
\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{LADD} & \
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle \quad \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e_1' + e_2' \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{RADD} & \
\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e_2' \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{ADD} & \
\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{LMUL} & \
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle \quad \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, e_1 * e_2 \rangle \rightarrow \langle \sigma', e_1' * e_2' \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{RMUL} & \
\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, n * e_2 \rangle \rightarrow \langle \sigma', n * e_2' \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{MUL} & \
\frac{p = m \times n}{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{ASSGN1} & \
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e_1' ; e_2 \rangle}
\end{align*}
\]

\[
\begin{align*}
\text{ASSGN} & \
\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle}
\end{align*}
\]