

CS 4110

Programming Languages & Logics

Lecture 4
Large-Step Semantics

29 August 2012



Review

So far we've:

- Defined a simple language of arithmetic expressions
- Formalized its semantics as a “small-step” relation:
 $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$ and $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$
- Proved some basic properties by induction

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Today we'll:

- Formalize its semantics as a “large-step” relation
- Prove the equivalence of these two semantics by induction

Large-Step Semantics

Idea: define a large-step relation that captures the *complete* evaluation of an expression.

Formally: define a relation \Downarrow of type:

$$\Downarrow \subseteq (\mathbf{Store} \times \mathbf{Exp}) \times (\mathbf{Store} \times \mathbf{Int})$$

Notation: write $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$ to indicate that $((\sigma, e), (\sigma', n)) \in \Downarrow$

Intuition: the expression e with store σ evaluates in one big step to the final store σ' and integer n .

Integers

$$\overline{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle}^{\text{Int}}$$

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle} \text{Var}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{Add}$$

Multiplication

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{Mul}$$

Assignment

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \text{ Assgn}$$

Large-Step Semantics

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle} \text{Int}$$

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle} \text{Var}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{Add}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{Mul}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \text{Assgn}$$

Example

Assume that $\sigma(\text{bar}) = 7$.

$$\frac{\frac{\frac{\langle \sigma', \text{foo} \rangle \Downarrow \langle \sigma', 3 \rangle}{\langle \sigma', \text{foo} \rangle \Downarrow \langle \sigma', 3 \rangle} \text{ Int} \quad \frac{\frac{\langle \sigma', \text{bar} \rangle \Downarrow \langle \sigma', 7 \rangle}{\langle \sigma', \text{bar} \rangle \Downarrow \langle \sigma', 7 \rangle} \text{ Var}}{\langle \sigma', \text{foo} * \text{bar} \rangle \Downarrow \langle \sigma', 21 \rangle} \text{ Mul}}{\langle \sigma, \text{foo} := 3 ; \text{foo} * \text{bar} \rangle \Downarrow \langle \sigma', 21 \rangle} \text{ Assgn}$$

Equivalence

Theorem (Equivalence of small-step and large-step)

$$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle \text{ if and only if } \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$$

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To streamline the proof, we'll use the following multi-step relation:

$$\frac{\overline{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle} \text{ Refl}}{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle} \quad \frac{\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle} \text{ Trans}$$

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Theorem (Equivalence of small-step and large-step)

$$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle \text{ if and only if } \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$$

Lemma

1. If $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$, then:

- ▶ $\langle \sigma, e + e_2 \rangle \rightarrow^* \langle \sigma', n + e_2 \rangle$
- ▶ $\langle \sigma, n_1 + e \rangle \rightarrow^* \langle \sigma', n_1 + n \rangle$
- ▶ $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
- ▶ $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$

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- ▶ $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
- ▶ $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$

2. If $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$ and $\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle$, then $\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle$

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- ▶ $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$

2. If $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$ and $\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle$, then $\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle$

3. If $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$ and $\langle \sigma'', e'' \rangle \Downarrow \langle \sigma', n \rangle$, then $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$