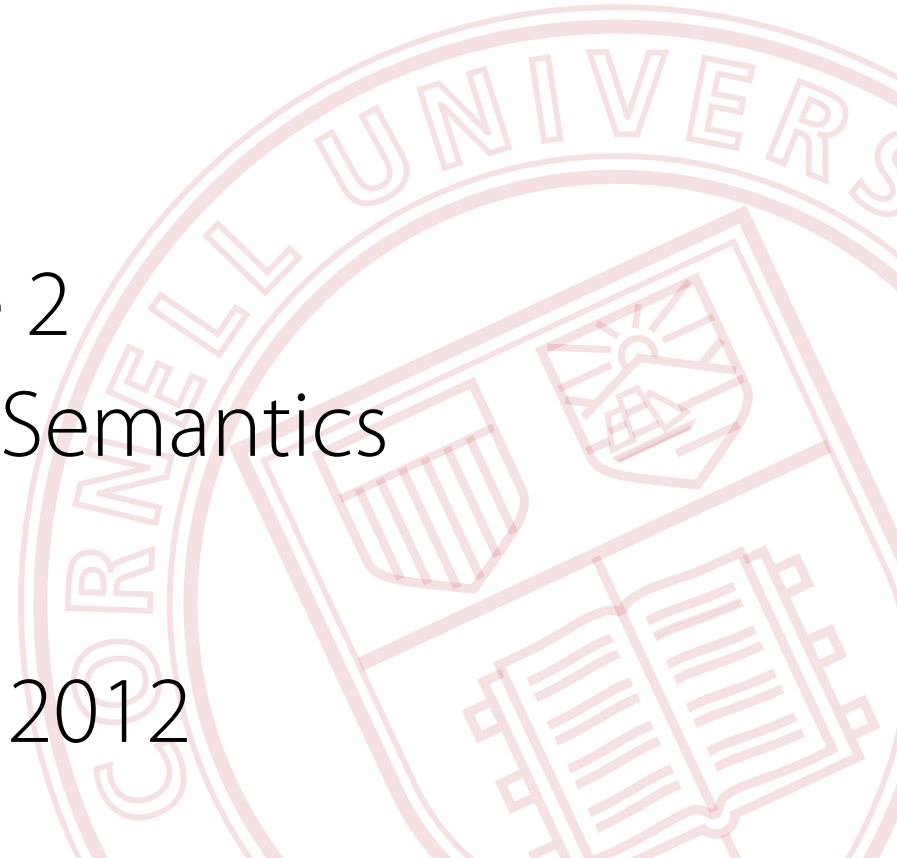
CS 4110

Programming Languages & Logics



Lecture 2
Introduction to Semantics

24 August 2012

Announcements

OCaml Demo

- 7-9pm tonight in Upson B7

Teaching Assistants

- Brittany Nkounkou
- Raghu Rajkumar

Homework #1

- Out: Monday, August 27th
- Due: Monday, September 3rd
- Distributed via CMS

Semantics

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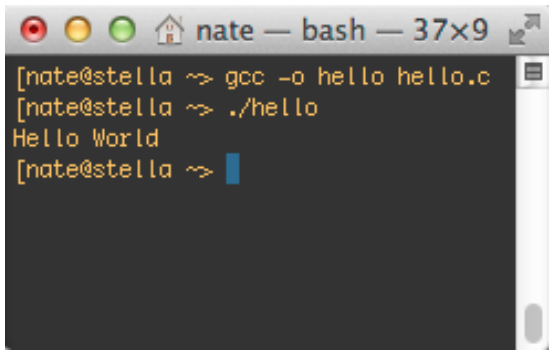
Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...



A6.7 Void

The (nonexistent) value of a `void` object may not be used in any way, and neither explicit nor implicit conversion to any non-void type may be applied. Because a void expression denotes a nonexistent value, such an expression may be used only where the value is not required, for example as an expression statement (§A9.2) or as the left operand of a comma operator (§A7.18).

An expression may be converted to type `void` by a cast. For example, a void cast documents the discarding of the value of a function call used as an expression statement.

`void` did not appear in the first edition of this book, but has become common since.

...but neither of these is a satisfactory solution.

Formal Semantics

Three Approaches

- Operational $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters

- Axiomatic $\vdash \{\phi\}\, e\, \{\psi\}$
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness

- Denotational: $[\![e]\!]$
  - Model program as mathematical objects
  - Useful for theoretical foundations

Arithmetic Expressions

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BNF Grammar:

$$e ::= x \mid n \mid e_1 + e_2 \mid e_1 * e_2 \mid x := e_1 \,;\, e_2$$

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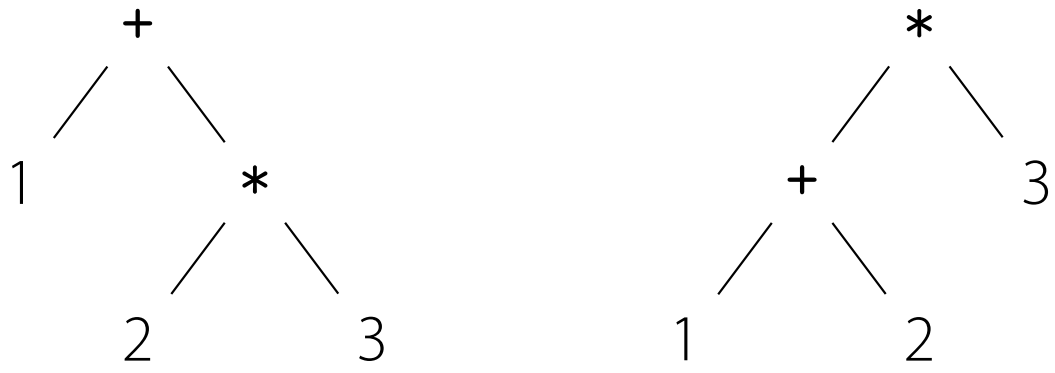

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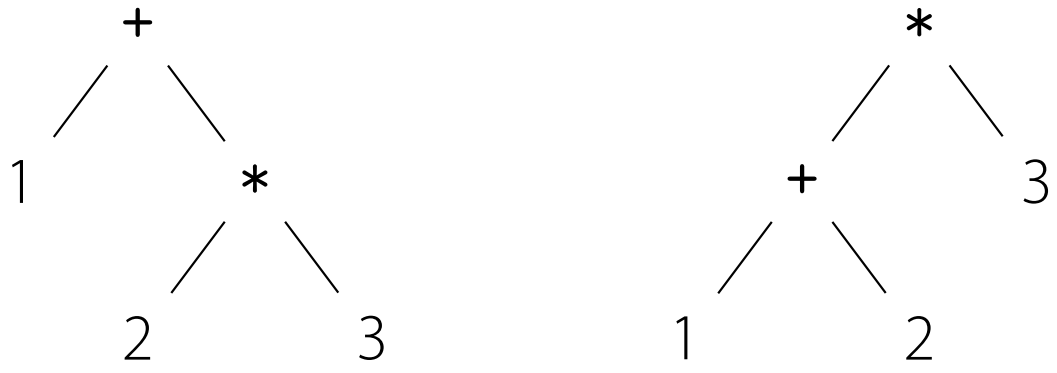

In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).

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OCaml:

```
type exp = Var of string
         | Int of int
         | Add of exp * exp
         | Mul of exp * exp
         | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var "foo", Int 1))

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Java:

```
abstract class Expr { }
class Var extends Expr { String name; .. }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; .. }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))

Quiz

- $7 + (4 * 2)$ evaluates to ...?

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The rest of this lecture will make these intuitions precise...

Mathematical Preliminaries

Binary Relations

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Some Important Relations

- empty – $\emptyset$
- total – $A \times B$
- identity on A – $\{(a, a) \mid a \in A\}$.
- composition $R; S$ – $\{(a, c) \mid \exists b.\, (a, b) \in R \wedge (b, c) \in S\}$

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The *image* of f is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. More formally: $\text{image}(f) \triangleq \{f(a) \mid a \in A\}$

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Given two functions $f : A \to B$ and $g : B \to C$, the composition of f and g is defined by: $(g \circ f)(x) = g(f(x))$ Note order!

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A function $f : A \to B$ is said to be *surjective* (or *onto*) if and only if the image of f is B.

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More formally,

$$\begin{aligned} \textbf{Store} &\triangleq \textbf{Var} \rightharpoonup \textbf{Int} \\ \textbf{Config} &\triangleq \textbf{Store} \times \textbf{Exp} \end{aligned}$$

Note that a store is a *partial* function from variables to integers.

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Intuitively, if facts above the line hold, then facts below the line hold. More formally, "$\longrightarrow$" is the smallest relation "closed" under the inference rules.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \longrightarrow \langle \sigma, n \rangle}\ \text{Var}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \longrightarrow \langle \sigma', e_1' + e_2 \rangle}\ \text{LAdd}$$

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Multiplication

$$\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 * e_2 \rangle \longrightarrow \langle \sigma', e_1' * e_2 \rangle}\ \text{LMul}$$

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$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \longrightarrow \langle \sigma, p \rangle}\ \text{Mul}$$

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$$\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 \,;\, e_2 \rangle \longrightarrow \langle \sigma', x := e_1' \,;\, e_2 \rangle} \text{ Assgn1}$$

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$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n \,;\, e_2 \rangle \longrightarrow \langle \sigma', e_2 \rangle}\ \text{Assgn}$$

Notation: $\sigma[x \mapsto n]$ maps x to n and otherwise behaves like σ

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$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \to \langle \sigma, n \rangle}\ \text{Var}$$

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