Linear Systems

What is the Matrix?

Outline

- Announcements:
  - Homework III: due Today. by 5, by e-mail
  - Ideas for Friday?
- Linear Systems Basics
- Matlab and Linear Algebra

Ecology of Linear Systems

- Linear Systems are found in every habitat:
  - simple algebra
  - solutions to ODEs & PDEs
  - statistics (especially, least squares)
- If you can formulate your problem as linear system, it will be easy to solve on a computer

"Standard Linear System"

- Simplest linear system: finding the equation of a line:
  - y=m*x + b
- The goal is to find m and b from observations of (x,y) pairs
- 2 points form a line, so we need two observations (x1, y1) & (x2,y2)

$$y_1 = mx_1 + b$$
$$y_2 = mx_2 + b$$

The 8th Grade Solution

- Solve for b in the first equation:

$$b = y_1 - mx_1$$

- Substitute this for b in 2nd equation & solve for m:

$$y_2 = mx_2 + y_1 - mx_1$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Put this into the equation for b above

The Sophomoric Solution

- Write the equations as a matrix problem

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} * \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Perform Gaussian Elimination on Matrix:

$$\left[\begin{array}{cc|c} 1 & 1/x_1 & y_1/x_1 \\ x_2 & 1 & y_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1/x_1 & y_1/x_1 \\ 0 & \frac{x_1-x_2}{x_1} & \frac{x_1y_2-x_2y_1}{x_1} \end{array}\right]$$

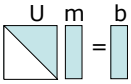
$$\left[\begin{array}{cc|c} 1 & 1/x_1 & y_1/x_1 \\ 0 & 1 & \frac{x_1y_2-x_2y_1}{x_1-x_2} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{y_1-y_2}{x_1-x_2} \\ 0 & \frac{x_1-x_2}{x_1} & \frac{x_1y_2-x_2y_1}{x_1-x_2} \end{array}\right]$$

Comparing methods

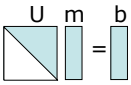
- Gaussian Elimination is a simpler algorithm
  - Easily generalizes to systems with more unknowns
- Gaussian Elimination is the starting point of much of numerical linear algebra

A Closer Look at GE (optional)

- For Am=y
  - GE reduces A to an upper triangular matrix U
  - y is modified--call it b



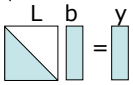
A Closer Look at GE (optional)



- last row corresponds to
  - $U_{n,n}*m_n=b_n$ --- can solve for m_n
- row above is
  - $U_{n-1,n-1}*m_{n-1}+ U_{n-1,n}*m_n =b_{n-1}$ --- can solve for m_{n-1}
- So, can work our way up rows to get m
  - Called "Back Substitution"

A Closer Look at GE (optional)

- but what is b?
  - b is solution to equation



- So, can work our way down rows to get b
  - Called "Forward Substitution"

A Closer Look at GE (optional)

- GE is equivalent to
  - LUm=y
- Implies that A=LU!
- So, GE works by finding lower & upper triangular matrices
  - triangular matrices are easy to work with
    - Forward/Backward substitution

Other Algorithms for Solving Linear Systems

- GE aka LU decomposition -- any A
- Cholesky Factorization -- symmetric, positive definite A
- Iterative solvers (conjugate gradients, GMRES)

Linear Systems in Matlab

- Linear systems are easy in Matlab
  - To solve Ax=b, type x=A\b
  - To solve x'A'=b', type x'=b'/A' (transposed)

More About \

- Matrix multiplication (*) is easy, fast
- Matrix "division" (\) is hard and computationally intensive
  - In general, performs GE with partial pivoting
  - But, \ is smart & looks closely at A for opportunities to speed up
    - If A is LT, just does back substitution
- If A is over-determined, A\b is the least-squares solution

Factorization

- Can explicitly factor A using LU:
  - [L,U]=lu(A)
  - useful if you have to solve A\b many times (different b each time)
    - To solve LUx=b: first solve Ly=b, then solve Ux=y
    - In Matlab: y=L\b; x=U\y;
- Other factorizations: chol, svd

What about A^{-1}?

- Matlab can compute A^{-1} using inv(A), but …
  - inv(A) is slower than lu(A)
  - There are numerical problems with inv(A)
- Rarely needed, use lu(A) or another factorization

$$
\begin{aligned}
s &= c'A^{-1}d \\
s &= c'U^{-1}L^{-1}d \\
Ly &= d \\
Ux &= y \\
s &= c'x
\end{aligned}
$$

Key Points on Linear Systems

- They're everywhere
- Easy to solve in Matlab (\)
- If possible, reuse factors from LU-decomposition
- Never use inv(A) for anything important
- Linear algebra (analytical and numerical) are highly recommended

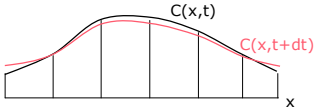
Advection-Diffusion

$$\frac{\partial C}{\partial t} = u\frac{\partial C}{\partial x} + k\frac{\partial^2 C}{\partial x^2}$$

- Model concentration of "contaminant" C
- Similar equations occur in
  - fluid dynamics
  - developmental biology
  - ecology

Numerical Solution

- We start with an initial distribution of C over the interval [0 1]
- Divide [0 1] into discrete points separated by dx



- C(x,t+dt) will depend on C(x), C(x-dx), & C(x+dx)

Numerical Solution

- replace partial derivatives with differences:

$$\frac{C_x^{t+dt} - C_x^t}{dt} = u\frac{C_{x+dx}^t - C_x^t}{dx} + k\frac{C_{x+dx}^{t+dt} - 2C_x^{t+dt} - C_{x-dx}^{t+dt}}{dx^2}$$

$$C_x^{t+dt} - \sigma(C_{x+dx}^{t+dt} - 2C_x^{t+dt} - C_{x-dx}^{t+dt}) = C_x^t + \lambda(C_{x+dx}^t - C_x^t)$$

$$\begin{bmatrix} -\sigma & (1+2\sigma) & -\sigma \end{bmatrix} * \begin{bmatrix} C_{x-dx}^{t+dt} \\ C_x^{t+dt} \\ C_{x+dx}^{t+dt} \end{bmatrix} = C_x^t + \lambda(C_{x+dx}^t - C_x^t)$$

- The solution of C(x,t+dt) depends on neighboring points

Numerical Solution

- This is a matrix problem
  - $A*C^{t+dt}=f(C^t)$
- Each grid point will have a row in matrix A
- All rows are the same except for first and last
  - We need to specify what happens at end points
  - Boundary conditions are a big problem
  - We'll use periodic BC's
    - C(0)=C(1), so first and last rows are:

$$\begin{bmatrix} (1+2\sigma) & -\sigma & \dots & -\sigma \end{bmatrix}$$
$$\begin{bmatrix} -\sigma & \dots & (1+2\sigma) & -\sigma \end{bmatrix}$$

Numerical Solution

- Algorithm:
  - build A
  - for j=1:n
    - build RHS=f(C^t)
    - C^{t+dt}=A\RHS ----- very time consuming!
  - end

Numerical Solution

- Same A used in each iteration
- Factor once:
  - build A
  - [L,U]=lu(A);%chol would be better..
  - for j=1:n
    - build RHS=f(C^t)
    - y=L\RHS; ---forward substitution
    - C^{t+dt}=U\y; ---back substitution
  - end

Sparse Matrices

- A is sparse
  - the only non-zero elements are immediately above, below, and on the diagonal
  - corners for periodic BC's
- Matlab has special sparse matrices
  - much less memory (don't need space for 0's)
  - faster to process
  - A=sparse(I,J,S) forms A s.t. A(I(j),J(j))=S(j)

AdvDiff1D.m

- Uses slightly more complicated procedure for advection known as "Lax-Wendroff" method
- Must specify
  - Initial concentration C0
  - parameters (u, k)
  - size of domain L
  - length of time T,
  - dx, dt
- Returns x, t, and C(x,t)