CS 381 **Introduction to Theory of Computing** Summer 2002
Prelim 1 Solutions June 7, 2002

10 points per problem.

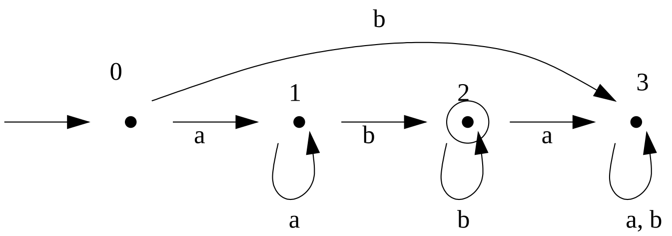
1. Consider the following DFAs with $\Sigma = \{a, b\}$. In each table, S indicates the start state and F indicates a final state.

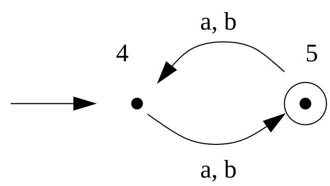
M_1	a	b
S0	1	3
1	1	2
F2	3	2
3	3	3

M_2	a	b
S4	5	5
F5	4	4

a. Draw the transition diagram for each DFA.

Solution:





b. Describe the set accepted by each. You may use words, patterns, or symbolic set notation.

Solution: From the diagram for the DFA on the left, we see that any string accepted must start with an a, followed by any number of a's, then have a b followed by any number of b's. So the language is $L(a^+b^+) = \{a^nb^m \mid n, m \geq 1\}$.

From the diagram on the right, any string accepted must first have a symbol, then be followed by an odd number of symbols, so the language accepted is

$$L((a+b)(aa+ab+ba+bb)^*) = \{x \in \{a,b\}^* \mid |x| \text{ is odd}\}.$$

c. Use the product construction to give the DFA accepting the **union** of these two languages. Represent your answer in tabular form. Be sure to show the start and final state(s).

Solution: The product construction for union requires that we use ordered pairs of the original states as new states with an ordered pair being a final state if either of its component states is final. The start state is the ordered pair consisting of the two original start states. In tabular form, this gives

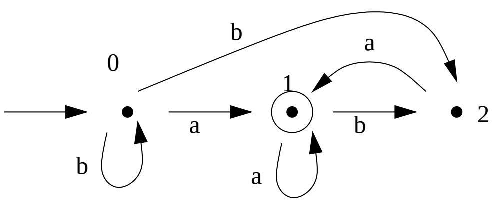
N	a	b
S(0, 4)	(1, 5)	(3, 5)
F(0, 5)	(1, 4)	(3, 4)
(1 ,4)	(1, 5)	(2, 5)
F(1, 5)	(1, 4)	(2, 4)
F(2, 4)	(3, 5)	(2, 5)
F(2, 5)	(3, 4)	(2, 4)
(3, 4)	(3, 5)	(3, 5)
F(3, 5)	(3, 4)	(3, 4)

2. Consider the following NFA:

N	a	b
S0	{1}	{0, 2}
F1	{1}	{2}
2	{1}	$\emptyset$

a. Draw the transition diagram for N.

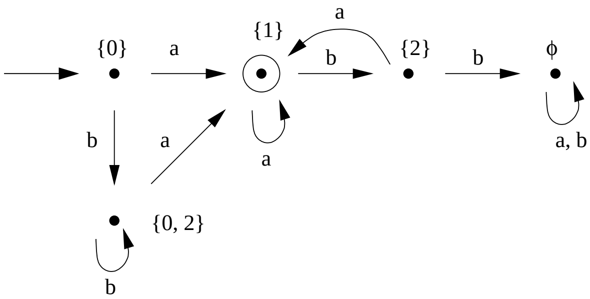
Solution:



b. Use the subset construction to give a DFA M with $L(M) = L(N)$. Express your results in tabular form. Be sure to show the start and final state(s).

Solution:

M	a	b
$\emptyset$	$\emptyset$	$\emptyset$
S{0}	{1}	{0, 2}
F{1}	{1}	{2}
{2}	{1}	$\emptyset$
F{0, 1}	{1}	{0, 2}
{0, 2}	{1}	{0, 2}
F{1, 2}	{1}	{2}
F{0, 1, 2}	{1}	{0, 2}



c. Draw the transition diagram for M. Omit inaccessible states.

Solution: See above.

3. Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA. Recall that $\hat{\delta}$ is defined recursively by

$$\hat{\delta}(p, \epsilon) = p$$
$$\hat{\delta}(p, xa) = \delta(\hat{\delta}(p, x), a)$$

for $p \in Q, x \in \Sigma^*, a \in \Sigma$. Use induction on $|y|$ to prove that for all $x, y \in \Sigma^*$ and all $p \in Q$,

$$\hat{\delta}(p, xy) = \hat{\delta}(\hat{\delta}(p, x), y).$$

Proof: For the basis case we have $y = \epsilon$. In this case, for $p \in Q$ and $x \in \Sigma^*$ we have

$$\begin{aligned}\hat{\delta}(p, xy) &= \hat{\delta}(p, x) \text{ since } y = \epsilon \\ &= \delta(\hat{\delta}(p, x), \epsilon) \text{ by definition of } \delta \\ &= \hat{\delta}(\hat{\delta}(p, x), y) \text{ by definition of } \hat{\delta}, y.\end{aligned}$$

For the induction hypothesis, we assume that $\hat{\delta}(p, xy) = \hat{\delta}(\hat{\delta}(p, x), y)$ for some string y.

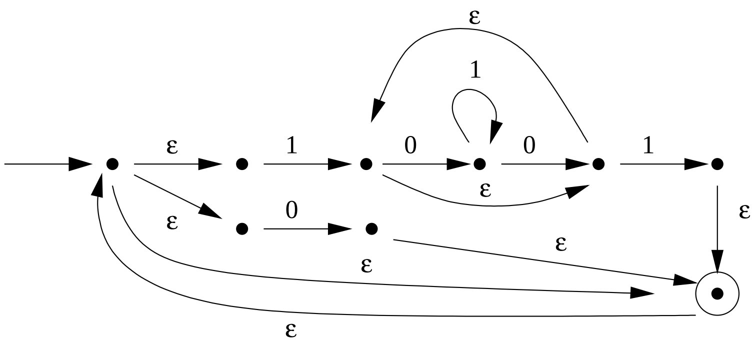
For the induction step, we have for any $a \in \Sigma$ that

$$\begin{aligned}\hat{\delta}(p, xya) &= \delta(\hat{\delta}(p, xy), a) \text{ by def of } \hat{\delta} \\ &= \delta(\hat{\delta}(\hat{\delta}(p, x), y), a) \text{ by induction hypothesis} \\ &= \delta(\hat{\delta}(q, y), a) \text{ where } q = \hat{\delta}(p, x) \\ &= \hat{\delta}(q, ya) \text{ by def of } \hat{\delta} \\ &= \hat{\delta}(\hat{\delta}(p, x), ya) \text{ by def of } q.\end{aligned}$$

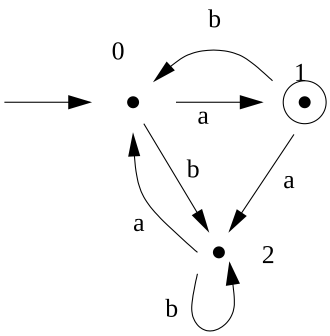
By induction, the claim is true for all strings y.

4. a. For the regular expression $\alpha = (0 + 1(01^*0)^*1)^*$, draw the transition diagram of an ϵ-NFA, N, so that $L(\alpha) = L(N)$.

Solution: There are many possible solutions. One is shown below, although it is not minimal.



b. Convert the following DFA to a regular expression using the recursive formula obtained by successively removing states of the DFA. For your solution, remove state 1 for the first recursive step, then either determine the remaining terms by "eyeballing" or else continue the recursion by removing state 2.



Solution: Removing state 1, the recursive formula gives

$$\alpha_{0,1}^{Q} = \alpha_{0,1}^{Q-\{1\}} + \alpha_{0,1}^{Q-\{1\}}(\alpha_{1,1}^{Q-\{1\}})^*$$

Note that we can leave off the final term in this case since the destination state and the state removed are the same. An examination of the diagram gives

$$\alpha_{0,1}^{Q} = (bb^*a)^*a + (bb^*a)^*a(ab^*(abb^*)^*aa + b(bb^*a)^*a)^*$$

There are other ways to write each term, but the general form should be the same.

5. a. Show that the set $A = \{rr \mid r \in \{0,1\}^*\}$ is not regular.

Solution: The most common error in this problem was to try to use the pumping lemma on a string with only one symbol or a string of the form $(01)^k$. With a string of this form and the proper choice of v, you can pump as many v's in as you like and stay in A.

To apply the contrapositive of the pumping lemma, first let $k \geq 0$, and choose $x = 1$, $y = 00^k$, $z = 100^k$. Then $xyz \in A$, and if $uvw = y$ with $|v| = m > 0$, then taking $i = 2$ we have

$$xuv^iwz = 10^{k+m+1}10^k,$$

and since $m > 0$, this string is not in A. Hence, A is not regular.

b. Let A be a regular set. Show that the set $R = \{x \mid xy \in A\}$ is a regular set.

Solution: Since A is regular, there is a DFA $M = (Q, \Sigma, \delta, s, F)$ with $L(M) = A$. Define a new DFA $N = (Q, \Sigma, \delta, s, F')$, where

$$F' = \{q \in Q \mid \exists y \in \Sigma^* \text{ so that } \hat{\delta}(q, y) \in F\}.$$

We need to show that $L(N) = R$. To do this, first suppose $x \in R$. Then there exists $y \in \Sigma^*$ so that $xy \in A$. By problem 3, we know that

$$\hat{\delta}(\hat{\delta}(s, x), y) = \hat{\delta}(s, xy),$$

which is contained in F by definition of accept for M. By the definition of F', we see that $\hat{\delta}(s, x)$ is in F', so by definition of accept, N accepts x, so $x \in L(N)$.

Next suppose $x \in L(N)$. Then $\hat{\delta}(s, x) \in F'$, so by definition of F' there exists $y \in \Sigma^*$ so that $\hat{\delta}(\hat{\delta}(s, x), y) \in F$, so again by problem 3 we have $\hat{\delta}(s, xy) \in F$, hence $xy \in A$. Thus $x \in R$. Since both subset relations hold, we have $R = L(N)$.