CS381 Fall 2000	**Handout 31**
November 10, 2000	

1. Reading: D. Kozen *Automata and Computability*, lectures 32, 33, 35
J. Hopcroft and J. Ullman *Introduction to Automata Theory, etc.*, section 8.4.

2. The main message of this lecture:

> **Today we are catching up with the proof of the Rice Theorem and give more examples on decidability and semidecidability.**

Comment 31.1. Does the Rice Theorem claim that everything about programs (Turing machines) is undecidable? Not at all! There are plenty of decidable properties of programs (cf. Examples 30.9). However, if a property of a machine M is in fact a property of an underlying r.e. set, then the Rice Theorem applies and this property is either trivial or undecidable. Here is a paradigmatic example.

Example 31.2. Which of the properties of a Turing machine M is decidable
a) M accepts 000,
b) M accepts 000 after making ≤ 1000000 steps?

Answer a) Undecidable, by the Rice Theorem.
b) Decidable: run M on 000 and reject if not accepted after 1000000 steps.
Why does the Rice Theorem apply to (a) but not apply to (b)? Note that M accepts 000 if and only if $000 \in \alpha = L(M)$, therefore (a) is equivalent to a property of an r.e. language α which is invariant with respect to a choice of of machine accepting α. The property (b) is NOT invariant with respect to a choice of M: for some r.e. set α (for example, $\alpha = \{000\}$ there are M and M' such that $\alpha = L(M) = L(M')$, but M satisfies (b) whereas M' does not! Therefore, (b) is not equivalent to any property of r.e. languages and Rice does not apply.

Example 31.3. The following properties of M are not decidable:
1. M accepts at least one string,
2. M accepts at least 381 distinct strings,
3. M accepts all strings,
4. M accepts a recursive set,
5. M accepts a nonrecursive set,
6. M accepts ϵ.

Proof. We will reformulate all those properties as properties of sets and thus place them all inside the scope of the Rice Theorem.
1. $\Leftrightarrow\ L(M) \neq \emptyset$,
2. $\Leftrightarrow\ |L(M)| \geq 381$,
3. $\Leftrightarrow\ L(M) = \Sigma^*$,
4. $\Leftrightarrow\ L(M)$ *is recursive*,
5. $\Leftrightarrow\ L(M)$ *is not recursive*,
6. $\Leftrightarrow\ \epsilon \in L(M)$.

Example 31.4. The following properties of M are not decidable:

1. M halts on at least one input,
2. $M(x)$ is total, i.e. M halts on every input,
3. M starting with ϵ ever prints 111 on the tape (cf. H& U, Example 8.4.).

Proof. Each of these problems requires a *reduction* to a problem to which the Rice Theorem is already applicable directly.

1. Let M be an arbitrary TM. Modify M to M' in such a way that whenever M reaches a reject state, M' enters an infinite loop without halt states. It is clear that

$$M \text{ accepts } x \;\Leftrightarrow\; M' \text{ accepts } x \Leftrightarrow\; M' \text{ halts on } x.$$

If (1) were decidable then the property $L(M) \neq \emptyset$ would be decidable too. Indeed, given M recast it to M' as above and decide whether M' halts on at least one input. If YES, then $L(M) \neq \emptyset$, if NO, then $L(M) = \emptyset$.

2. Recast M to M' as above. Then M accepts x if and only if M' halts on x, therefore $L(M) = \Sigma^*$ if and only if M' halts on every input. Therefore, decidability of (2) would yield decidability of $L(M) = \Sigma^*$, which is impossible by 31.3(3).

3. For each M construct M' that emulates M on an empty tape but uses 00 to encode a 0 and 01 to encode a 1. This can be easily done in such a way that M' never prints 111 on the tape. Now further modify M' to M'' such that if M' accepts, then M'' prints 111 and then accepts. Thus M'' prints 111 if and only if M accepts ϵ, which in undecidable, by the Rice Theorem. Again, any decision procedure for (3) would give a decision algorithm for 'M accepts ϵ', which is impossible.

Example 31.5. The following properties of M are semidecidable (r.e.):

1. M accepts ϵ,
2. M accepts at least one string,
3. M accepts at least 381 distinct strings.

Proof. 1. $M_i(\epsilon)$ accepts $\;\Leftrightarrow\;$ $U(i, \epsilon)$ accepts.

2. A positive test on M: a) put $n = 1$, b)run n steps of M on each of the inputs of length $\leq n$ (a finite set) until M accepts (and then test itself accepts), c) increment n and goto b). It is clear that if M accepts some x after n steps, then nth iteration of the test accepts M.

3. Do as in 2, but accept M only after M accepts 381 distinct x's.

Example 31.6. The following properties of M are not r.e.:

1. M does not accept ϵ,
2. M accepts nothing,
3. M accepts less than 381 distinct strings.

Proof. By 31.3, all of those properties are undecidable. By 31.5, their compliments are r.e.. Therefore, by the Post Theorem, they are all not r.e.

Homework problems. Kozen p.310, No.1,2; p.343, No.107