CS381 Fall 2000 **Handout 30**
November 8, 2000

1. Reading: D. Kozen *Automata and Computability*, lectures 29, 32, 34
J. Hopcroft and J. Ullman *Introduction to Automata Theory, etc.*, section 8.4.

2. The main message of this lecture:

Every nontrivial property of languages accepted by Turing machines is undecidable.

Definition 30.1. A set A is called *recursively enumerable (r.e.)* (or *semidecidable*) if A is accepted by a Turing Machine ($A = L(M)$ for some TM M). A set A is called *recursive*, (or *decidable*) if $A = L(M)$ for some **total** Turing Machine M. Obviously, every recursive set is r.e.. The same terminology applies to *properties*, since sets and properties are closely related: a property of strings specifies a set of strings satisfying this property.

Theorem 30.2. *Recursive sets are closed under unions, intersections, and complementations.*

Proof. Let $A = L(M_1)$ and $B = L(M_2)$ for some total TMs M_1 and M_2. The set $A \cup B$ is accepted by a total Turing machine that runs M_1 and M_2 and accepts when either of them accept. To accept $\sim A$ swap accept/reject states in M_1. To accept $A \cap B$ use the identity $A \cap B = \sim(\sim A \cup \sim B)$ and the previous results concerning $\cup$ and $\sim$.

Theorem 30.3. *R.e. sets are closed under unions and intersections.*

Proof. For union use the same construction as in 30.2. For intersection, consider $A = L(M_1)$ and $B = L(M_2)$ for some TMs M_1 and M_2. The set $A \cap B$ is accepted by the following program: run M_1 and M_2 on x and accept only when both M_1 and M_2 accept.

Theorem 30.4. (Post Theorem) *A set A is recursive if and only if both A and $\sim A$ are r.e..*

Proof. If A is recursive then, by 30.2, $\sim A$ is also recursive and thus both A and $\sim A$ are r.e.. Let now both A and $\sim A$ are r.e., $A = L(M_1)$ and $\sim A = L(M_2)$. Consider M that runs concurrently M_1 and M_2, accepts, when M_1 accepts, and rejects, when M_2 accepts. By the assumptions that $A = L(M_1)$ and $\sim A = L(M_2)$, on every input x either M_1 accepts (if $x \in A$), or M_2 accepts on (if $x \in \sim A$). Therefore, M is a total TM accepting A.

Theorem 30.5. *R.e. sets are not closed under complementations.*

Proof. Consider the halting set $K = \{i \mid U(i,i) \textit{ terminates}\}$. K is semidecidable (r.e.). Indeed, a TM accepting K works as follows: run $U(i,i)$ and accept when it terminates. Now, if $\sim K$ were r.e., then, by 29.5, K should be recursive (i.e. decidable) which contradicts theorem 29.5.

Definition 30.6. Let $P(A)$ be a property of r.e. sets A (for example, 'A is finite'). A property P is *nontrivial*, if it is neither universally true nor universally false; that is, there is at least one r.e. set that satisfies P and at least one that does not. An *index set* I_P of the property P is the set of all indices i such that the set M_i satisfies the property P. A property P is called decidable, if I_P is decidable. In other words, a property P of r.e. sets is decidable if there is a algorithm that given a Turing Machine code i decides whether M_i satisfies the property P.

Theorem 30.7. (Rice Theorem) *Every nontrivial property of r.e. sets is undecidable.*

Proof. Let P be a nontrivial property (set) of r.e. sets. The empty language $\emptyset$ is either in P or in $\sim P$. Let us assume the latter, i.e. $\emptyset \in \sim P$, the case $\emptyset \in P$ is symmetric. Since P is nontrivial, there is a set $A \in P$. Since A is r.e., $A = L(M)$ for some TM M. Consider a TM $H(m, x)$ that accepts if both $U(m, m)$ halts and $M(x)$ accepts, and loops otherwise. For each m we define $H_m(x)$ as $H(m, x)$, i.e. H_m on x first types $m\sharp x$, and then runs H on this input. It is clear that $L(H_m)$ is either $A \in P$ (if $m \in K$, where K is the halting set $\{i \mid U(i,i) \text{ } halts\}$), or $L(H_m) = \emptyset$ (if $m \notin K$). By the parameter property of U, there is a total computable function φ such that $H(m, x) \cong U(\varphi(m), x)$. Since $U(\varphi(m), x) \cong M_{\varphi(m)}(x)$, $\varphi(m)$ is an index of the set $L(H_m)$. From all the above it follows that

$$m \in K \ \Leftrightarrow\ L(H_m) \in P \ \Leftrightarrow\ \varphi(m) \in I_P.$$

If I_P were decidable, then we would have a decision algorithm for K, which is impossible.

Examples 30.8. Properties of TM M:
$L(M)$ is finite,
$L(M)$ is regular,
$L(M)$ is nonempty,
$L(M)$ is a CFL,
M accepts the empty string ϵ, etc.
are all undecidable. Indeed, each of those properties is a nontrivial property of r.e. sets and thus is undecidable.

Examples 30.9. Properties of TM Ms
M has at least 481 states,
M takes more than 481 steps on input x,
M takes more than 481 steps on some input,
M takes more than 481 steps on all inputs,
are decidable (cf. Lecture 32 from Kozen). The Rice theorem does not apply here since none of those properties can be identified with a nontrivial property of r.e. sets.

Homework problems. Kozen p.344, No.111, 112.