Numbers and Arithmetic

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CS 3410

Computer Science

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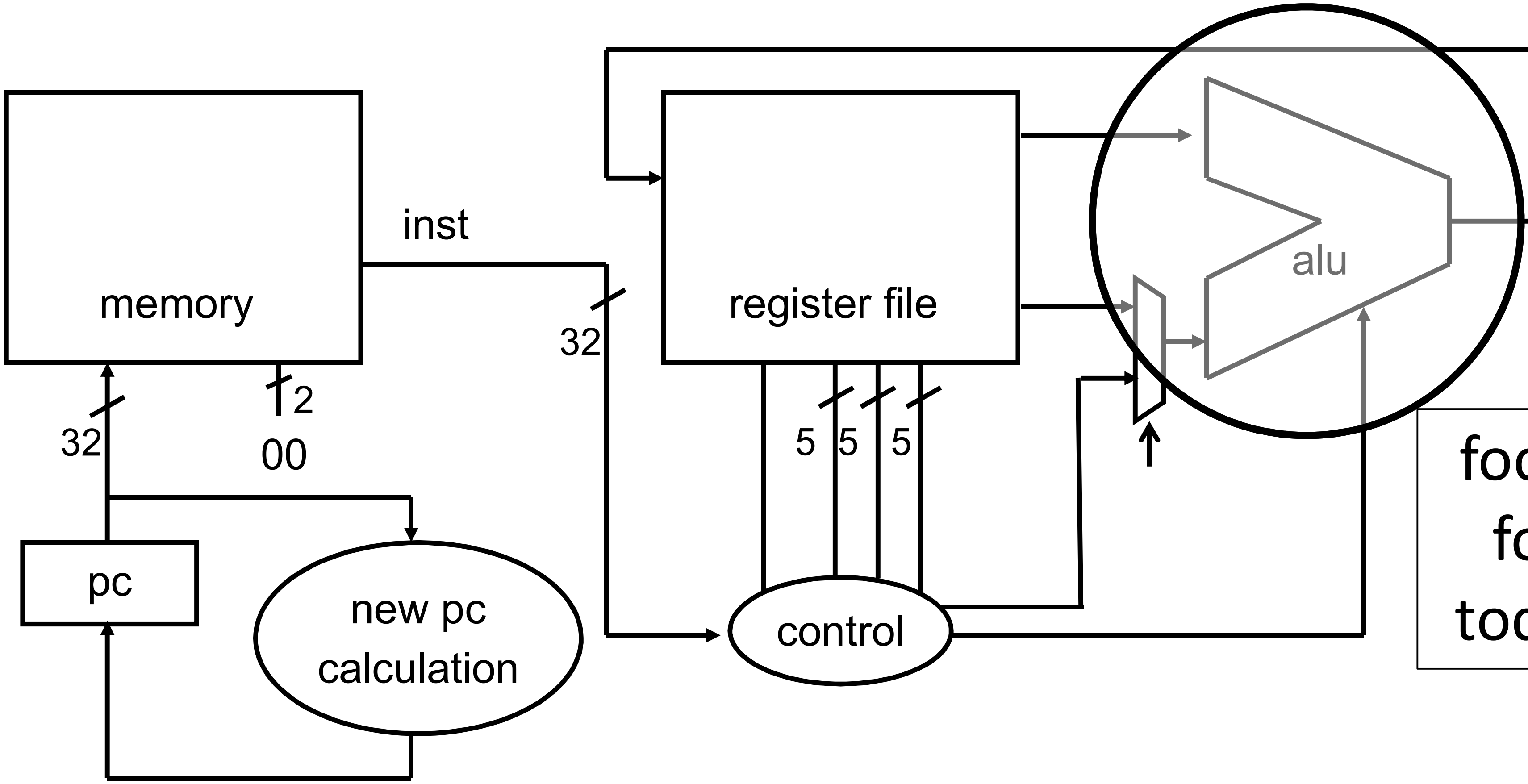
The slides are the product of many rounds of teaching CS 3410 by Professors Weatherspoon, Bala, Bracy, and Sirer.

Big Picture: Building a Processor



focus for today

Simplified Single-cycle processor

Goals for Today

Binary Operations

- Number representations
- One-bit and four-bit adders
- Negative numbers and two's compliment
- Addition (two's compliment)
- Subtraction (two's compliment)

Number Representations

Recall: Binary

- Two symbols (base 2): true and false; 1 and 0
- Basis of Logic Circuits and all digital computers

So, how do we represent numbers in *Binary* (base 2)?

Number Representations

Recall: Binary

- Two symbols (base 2): true and false; 1 and 0
- Basis of Logic Circuits and all digital computers

So, how do we represent numbers in *Binary* (base 2)?

- We can represent numbers in Decimal (base 10).
  - E.g. $\underset{10^2}{\underline{6}}\ \underset{10^1}{\underline{3}}\ \underset{10^0}{\underline{7}}$
- Can just as easily use other bases
  - Base 2 — Binary $\underset{2^9}{\underline{1}}\ \underset{2^8}{\underline{0}}\ \ \underset{2^7}{\underline{0}}\ \underset{2^6}{\underline{1}}\ \underset{2^5}{\underline{1}}\ \underset{2^4}{\underline{1}}\ \ \underset{2^3}{\underline{1}}\ \underset{2^2}{\underline{1}}\ \underset{2^1}{\underline{0}}\ \underset{2^0}{\underline{1}}$
  - Base 8 — Octal 0o $\underset{8^3}{\underline{1}}\ \underset{8^2}{\underline{1}}\ \underset{8^1}{\underline{7}}\ \underset{8^0}{\underline{5}}$
  - Base 16 — Hexadecimal 0x $\underset{16^2}{\underline{2}}\ \underset{16^1}{\underline{7}}\ \underset{16^0}{\underline{d}}$

Number Representations

Recall: Binary

- Two symbols (base 2): true and false; 1 and 0
- Basis of Logic Circuits and all digital computers

So, how do we represent numbers in *Binary* (base 2)?

- We can represent numbers in Decimal (base 10).
  - E.g. $\underset{10^2}{\underline{6}}\ \underset{10^1}{\underline{3}}\ \underset{10^0}{\underline{7}}$ $6 \cdot 10^2 + 3 \cdot 10^1 + 7 \cdot 10^0 = 637$

- Can just as easily use other bases
  - Base 2 — Binary $1 \cdot 2^9 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^0 = 637$
  - Base 8 — Octal $1 \cdot 8^3 + 1 \cdot 8^2 + 7 \cdot 8^1 + 5 \cdot 8^0 = 637$
  - Base 16 — Hexadecimal $2 \cdot 16^2 + 7 \cdot 16^1 + d \cdot 16^0 = 637$
    $2 \cdot 16^2 + 7 \cdot 16^1 + 13 \cdot 16^0 = 637$

Number Representations: Activity #1 Counting

How do we count in different bases?

- Dec (base 10) Bin (base 2) Oct (base 8) Hex (base 16)

Dec (base 10)	Bin (base 2)	Oct (base 8)	Hex (base 16)
0	0	0	**0**
1	1	1	**1**
2	10	2	**2**
3	11	3	**3**
4	100	4	**4**
5	101	5	**5**
6	110	6	**6**
7	111	7	**7**
8	1000	10	**8**
9	1001	11	**9**
10	1010	12	**a**
11	1011	13	**b**
12	1100	14	**c**
13	1101	15	**d**
14	1110	16	**e**
15	1111	17	**f**
16	1 0000	20	10
17	1 0001	21	11
18	1 0010	22	12
.	.	.	.
.	.	.	.
99			
100			

0b 1111 1111 = ?

0b 1 0000 0000 = ?

0o 77 = ?

0o 100 = ?

0x ff = ?

0x 100 = ?

Number Representations: Activity #1 Counting

How do we count in different bases?

- Dec (base 10) Bin (base 2) Oct (base 8) Hex (base 16)

Dec (base 10)	Bin (base 2)	Oct (base 8)	Hex (base 16)
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	a
11	1011	13	b
12	1100	14	c
13	1101	15	d
14	1110	16	e
15	1111	17	f
16	1 0000	20	10
17	1 0001	21	11
18	1 0010	22	12
.	.	.	.
.	.	.	.
99			
100			

0b 1111 1111 = 255

0b 1 0000 0000 = 256

0o 77 = 63

0o 100 = 64

0x ff = 255

0x 100 = 256

Number Representations

How to convert a number between different bases?

Base conversion via repetitive division

- Divide by base, write remainder, move left with quotient

- $637 \div 8 = 79$ remainder 5 lsb (least significant bit)
- $79 \div 8 = 9$ remainder 7
- $9 \div 8 = 1$ remainder 1
- $1 \div 8 = 0$ remainder 1 msb (most significant bit)

$637 = 0o\ \underset{msb}{1}17\underset{lsb}{5}$

Number Representations

Convert a base 10 number to a base 2 number

Base conversion via repetitive division

- Divide by base, write remainder, move left with quotient

lsb (least significant bit)

- $637 \div 2 = 318$ remainder 1
- $318 \div 2 = 159$ remainder 0
- $159 \div 2 = 79$ remainder 1
- $79 \div 2 = 39$ remainder 1
- $39 \div 2 = 19$ remainder 1
- $19 \div 2 = 9$ remainder 1
- $9 \div 2 = 4$ remainder 1
- $4 \div 2 = 2$ remainder 0
- $2 \div 2 = 1$ remainder 0
- $1 \div 2 = 0$ remainder 1

msb (most significant bit)

637 = $_{\text{msb}}$10 0111 1101$_{\text{lsb}}$ (can also be written as 0b10 0111 1101)

Clicker Question!

Convert the number 657_{10} to base 16
What is the least significant digit of this number?

a) D
b) F
c) 0
d) 1
e) 11

Clicker Question!

Convert the number 657_{10} to base 16
What is the least significant digit of this number?

a) D
b) F
c) 0
d) 1
e) 11

Number Representations

Convert a base 10 number to a base 16 number

Base conversion via repetitive division

- Divide by base, write remainder, move left with quotient
- $657 \div 16 = 41$ remainder 1 lsb
- $41 \div 16 = 2$ remainder 9
- $2 \div 16 = 0$ remainder 2 msb

Thus, 637 = 0x291

Number Representations

Convert a base 10 number to a base 16 number

Base conversion via repetitive division

- Divide by base, write remainder, move left with quotient
- $637 \div 16 = 39$ remainder 13 lsb
- $39 \div 16 = 2$ remainder 7
- $2 \div 16 = 0$ remainder 2 msb

<u>dec</u>	=	<u>hex</u>	=	<u>bin</u>
10	=	0xa	=	1010
11	=	0xb	=	1011
12	=	0xc	=	1100
13	=	0xd	=	1101
14	=	0xe	=	1110
15	=	0xf	=	1111

637 = 0x 2 7 13 = ?

Thus, 637 = 0x27d

Number Representations

How to convert a number between different bases?

Base conversion via repetitive division

- Divide by base, write remainder, move left with quotient

- $637 \div 10 = 63$ remainder $\boxed{7}$ lsb (least significant bit)
- $63 \div 10 = 6$ remainder $\boxed{3}$
- $6 \div 10 = 0$ remainder $\boxed{6}$ msb (most significant bit)

Number Representations

Convert a base 2 number to base 8 (oct) or 16 (hex)

Binary to Hexadecimal

- Convert each nibble (group of four bits) from binary to hex
- A nibble (four bits) ranges in value from 0…15, which is one hex digit
  - Range: 0000…1111 (binary) => 0x0 …0xF (hex) => 0…15 (decimal)
- E.g. 0b 10   0111   1101

  2   7   d   → 0x27d

  - Thus, 637 = 0x27d = 0b10 0111 1101

Binary to Octal

- Convert each group of three bits from binary to oct
- Three bits range in value from 0…7, which is one octal digit
  - Range: 0000…1111 (binary) => 0x0 …0xF (hex) => 0…15 (decimal)
- E.g. 0b1 001   111   101

  1   1   7   5   → 0o 1175

  - Thus, 637 = 0o1175 = 0b10 0111 1101

Number Representations Summary

We can represent any number in any base

- Base 10 – Decimal

$$\underset{10^2}{\underline{6}}\ \underset{10^1}{\underline{3}}\ \underset{10^0}{\underline{7}}$$

$$6\cdot10^2 + 3\cdot10^1 + 7\cdot10^0 = 637$$

- Base 2 — Binary

$$\underset{2^9}{\underline{1}}\ \underset{2^8}{\underline{0}}\quad \underset{2^7}{\underline{0}}\ \underset{2^6}{\underline{1}}\ \underset{2^5}{\underline{1}}\ \underset{2^4}{\underline{1}}\quad \underset{2^3}{\underline{1}}\ \underset{2^2}{\underline{1}}\ \underset{2^1}{\underline{0}}\ \underset{2^0}{\underline{1}}$$

$$1\cdot2^9+1\cdot2^6+1\cdot2^5+1\cdot2^4+1\cdot2^3+1\cdot2^2+1\cdot2^0 = 637$$

- Base 8 — Octal

$$0o\ \underset{8^3}{\underline{1}}\ \underset{8^2}{\underline{1}}\ \underset{8^1}{\underline{7}}\ \underset{8^0}{\underline{5}}$$

$$1\cdot8^3 + 1\cdot8^2 + 7\cdot8^1 + 5\cdot8^0 = 637$$

- Base 16 — Hexadecimal

$$0x\ \underset{16^2}{\underline{2}}\ \underset{16^1}{\underline{7}}\ \underset{16^0}{\underline{d}}$$

$$2\cdot16^2 + 7\cdot16^1 + d\cdot16^0 = 637$$

$$2\cdot16^2 + 7\cdot16^1 + 13\cdot16^0 = 637$$

Achievement Unlocked!

There are 10 types of people in the world:

Those who understand binary

And those who do not

And those who know this joke was written in base 2

Takeaway

Digital computers are implemented via logic circuits and thus represent *all* numbers in binary (base 2).

We (humans) often write numbers as decimal and hexadecimal for convenience, so need to be able to convert to binary and back (to understand what the computer is doing!).

Today’s Lecture

Binary Operations

- Number representations
- One-bit and four-bit adders
- Negative numbers and two’s compliment
- Addition (two’s compliment)
- Subtraction (two’s compliment)

Next Goal

Binary Arithmetic: Add and Subtract two binary numbers

Binary Addition

How do we do arithmetic in binary?

```
  1
  183
+ 254
-----
  437
```

Addition works the same way regardless of base

- Add the digits in each position
- Propagate the carry

Carry-in

Carry-out

```
   111
   001110
 + 011100
 --------
   101010
```

Unsigned binary addition is pretty easy

- Combine two bits at a time
- Along with a carry

Binary Addition

How do we do arithmetic in binary?

$$
\begin{array}{r}
1 \\
183 \\
+\ 254 \\
\hline
437
\end{array}
$$

Addition works the same way regardless of base

- Add the digits in each position
- Propagate the carry

$$
\begin{array}{r}
111 \\
001110 \\
+\ 011100 \\
\hline
101010
\end{array}
$$

Unsigned binary addition is pretty easy

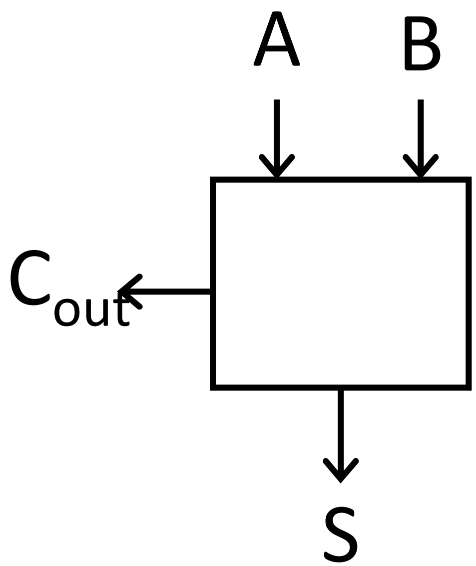
- Combine two bits at a time
- Along with a carry

Binary Addition

Binary addition requires

- Add of ***two bits*** PLUS ***carry-in***
- Also, ***carry-out*** if necessary

1-bit Adder



Half Adder

- Adds two 1-bit numbers
- Computes 1-bit result and 1-bit carry
- No carry-in

A	B	C_{out}	S
0	0		
0	1		
1	0		
1	1		

Clicker Question

What is the equation for C_{out}?

a) $A + B$

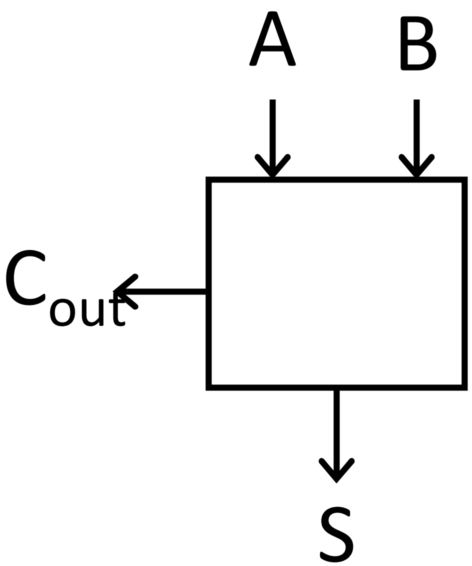
b) AB

c) $A \oplus B$

d) $A + !B$

e) $!A!B$

1-bit Adder



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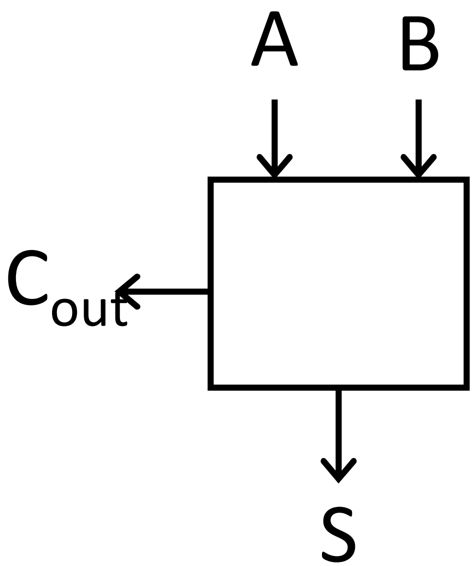
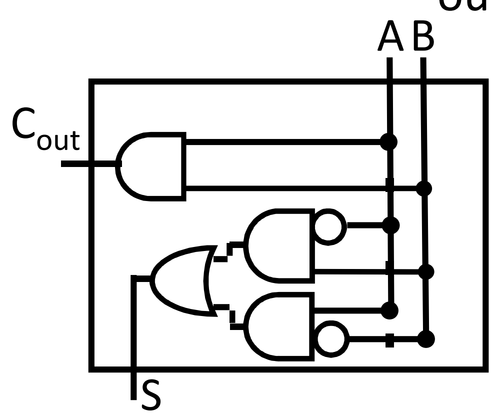


Half Adder

- Adds two 1-bit numbers
- Computes 1-bit result and 1-bit carry
- No carry-in

A	B	C_{out}	S
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- $S = \overline{A}B + A\overline{B}$
- $C_{out} = AB$



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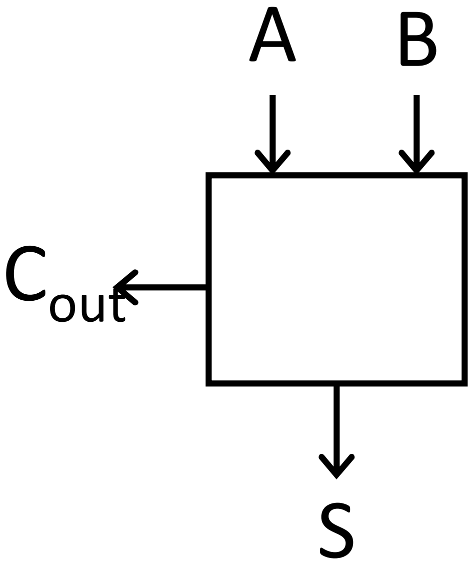
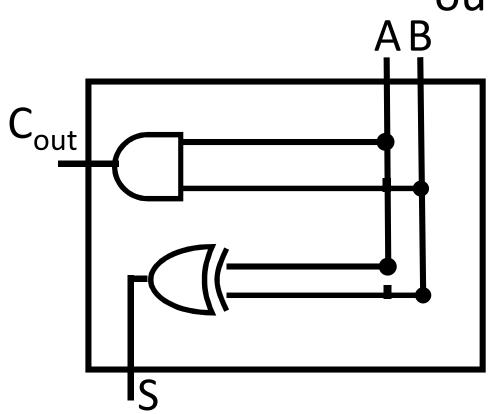


Half Adder

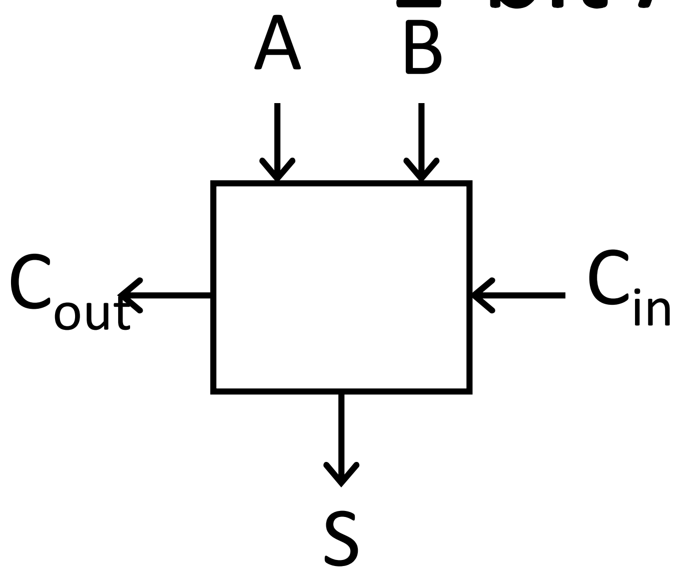
- Adds two 1-bit numbers
- Computes 1-bit result and 1-bit carry
- No carry-in

A	B	C_{out}	S
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- $S = \overline{A}B + A\overline{B} = A \oplus B$
- $C_{out} = AB$



1-bit Adder with Carry



Full Adder

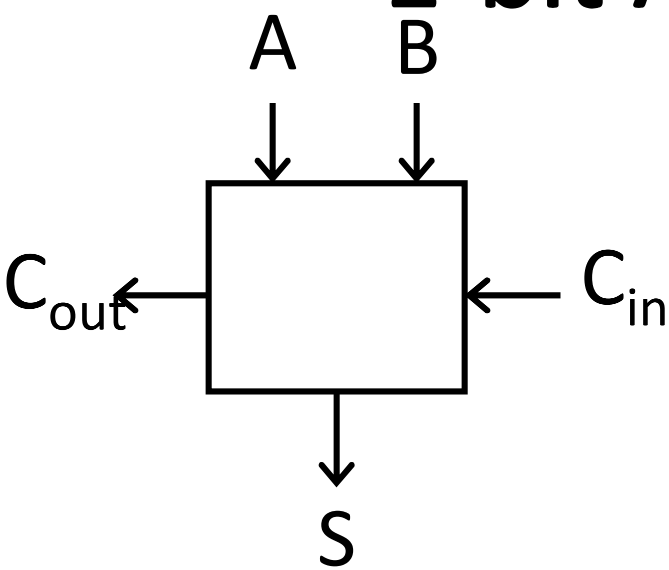
- Adds three 1-bit numbers
- Computes 1-bit result and 1-bit carry
- Can be cascaded

A	B	C_{in}	C_{out}	S
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Now You Try:

1. Fill in Truth Table
2. Create Sum-of-Product Form
3. Minimization the equation
   1. Karnaugh Maps (*coming soon!*)
   2. Algebraic minimization
4. Draw the Logic Circuits

1-bit Adder with Carry



Full Adder

- Adds three 1-bit numbers
- Computes 1-bit result and 1-bit carry
- Can be cascaded

A	**B**	**C_{in}**	**C_{out}**	**S**
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Clicker Question

What is the equation for C_{out}?

a) $A + B + C_{in}$

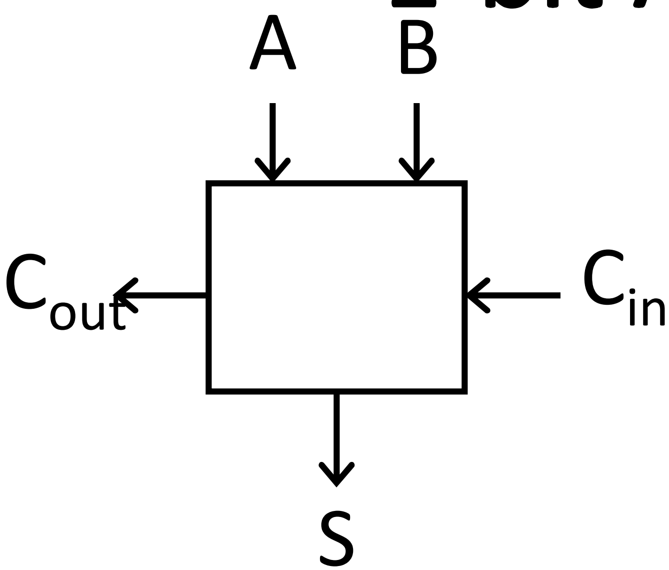
b) $!A + !B + !\ C_{in}$

c) $A \oplus B \oplus C_{in}$

d) $AB + AC_{in} + BC_{in}$

e) ABC_{in}

1-bit Adder with Carry



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A	B	C_{in}	C_{out}	S
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
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1	1	0		
1	1	1		

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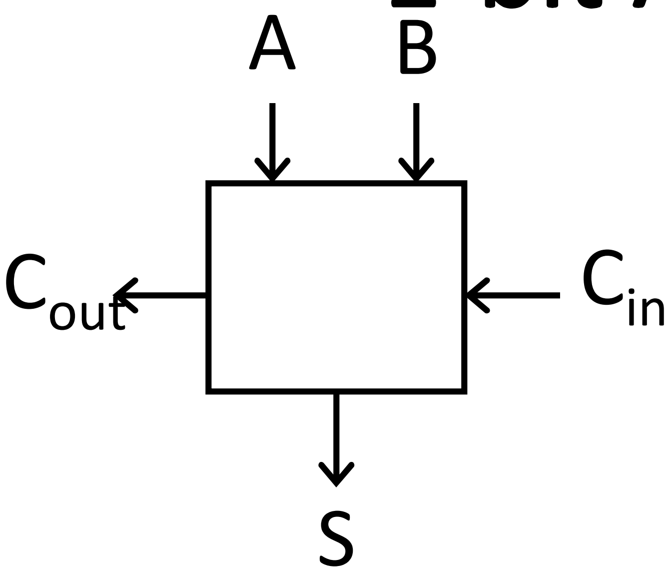
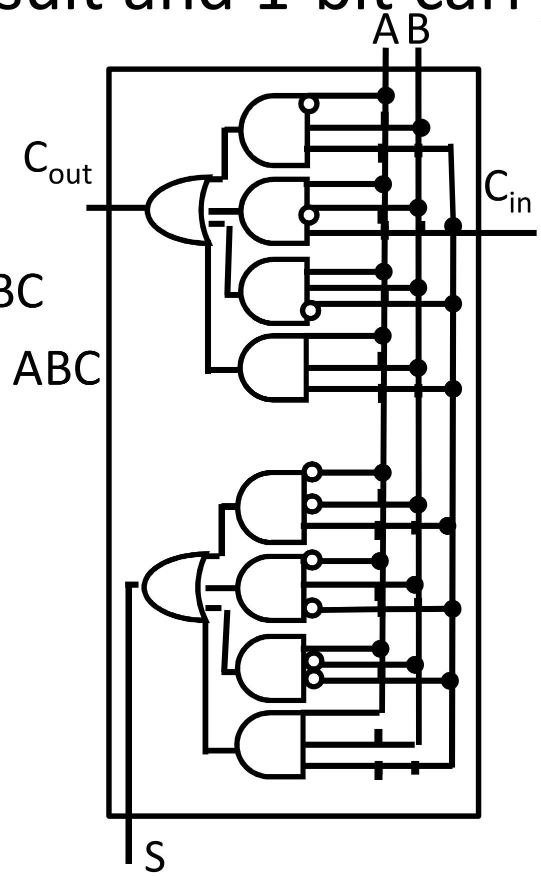
1-bit Adder with Carry



A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

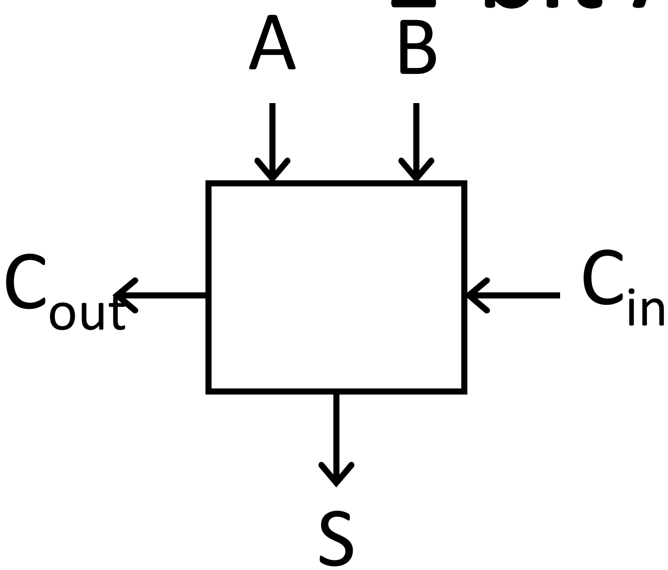
Full Adder

- Adds three 1-bit numbers
- Computes 1-bit result and 1-bit carry
- Can be cascaded



- $S = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}\bar{C} + ABC$
- $C_{out} = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$

1-bit Adder with Carry



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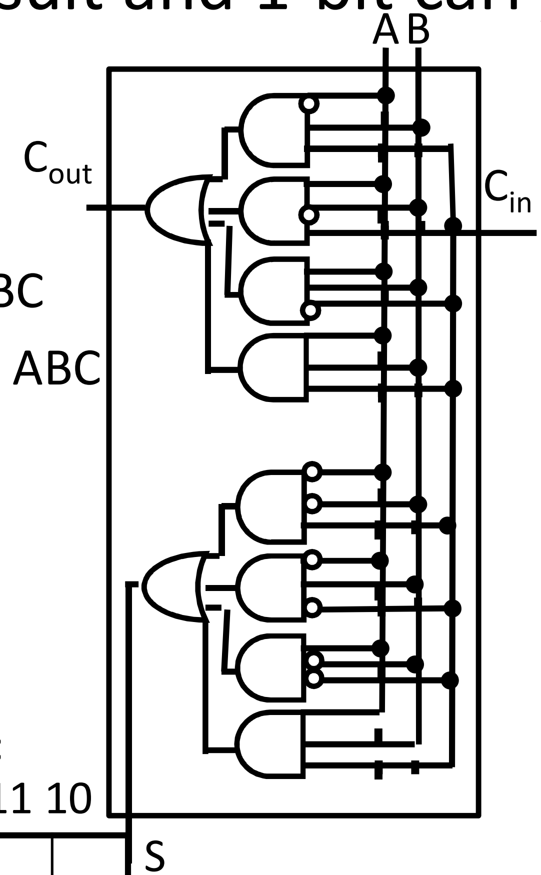


- $S = \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + ABC$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$

A	**B**	**C_{in}**	**C_{out}**	**S**
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

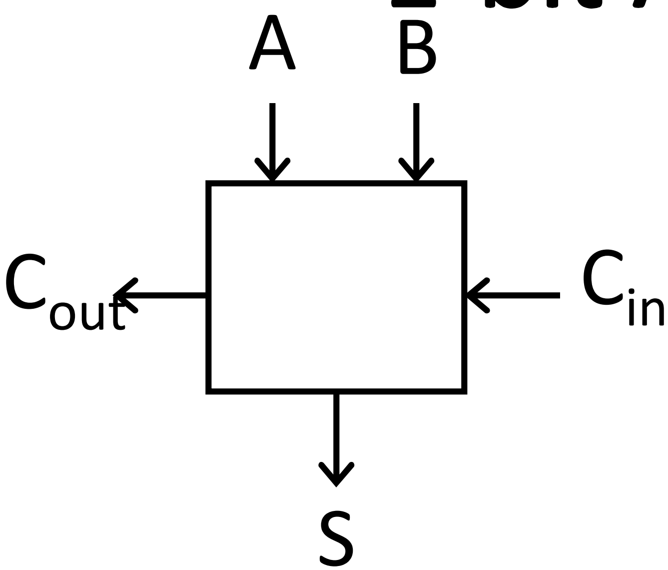
S

C_{in} \ AB	00	01	11	10
0				
1				

C_{out}

C_{in} \ AB	00	01	11	10
0				
1				

1-bit Adder with Carry



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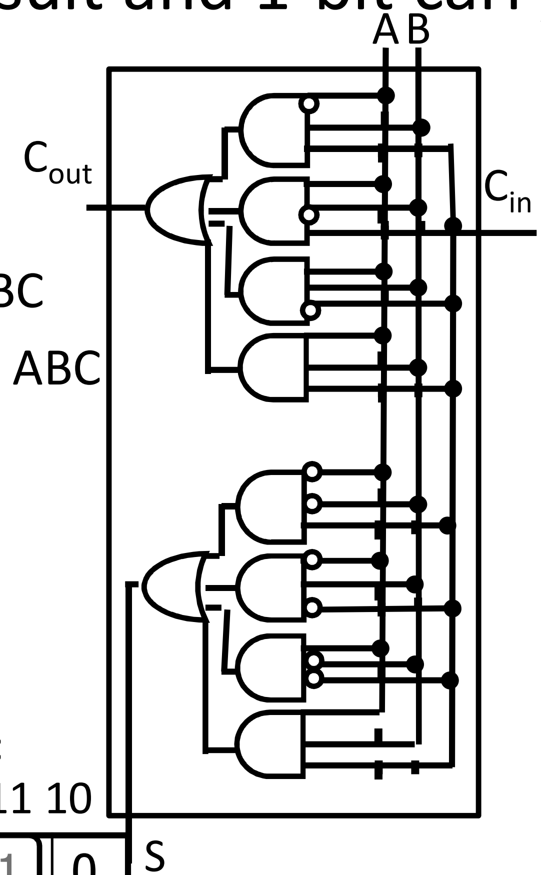


- $S = \overline{A}\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\overline{C} + ABC$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$
- $C_{out} = AB + AC + BC$

A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

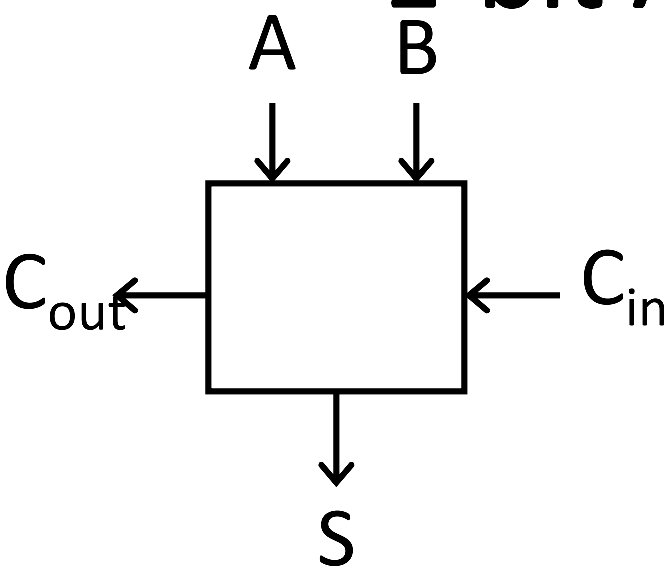
S

C_{in} \ AB	00	01	11	10
0	0	1	0	1
1	1	0	1	0

C_{out}

C_{in} \ AB	00	01	11	10
0	0	0	1	0
1	0	1	1	1

1-bit Adder with Carry



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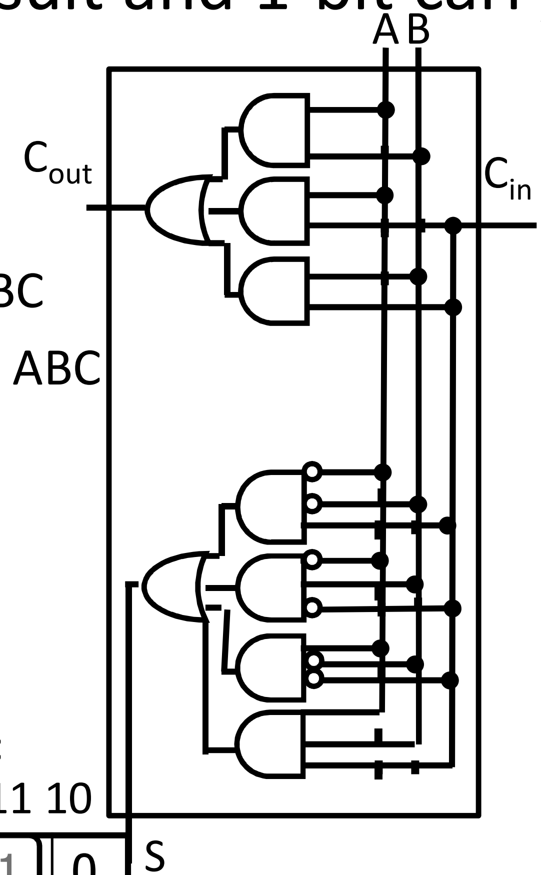


- $S = \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + ABC$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$
- $C_{out} = AB + AC + BC$

A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

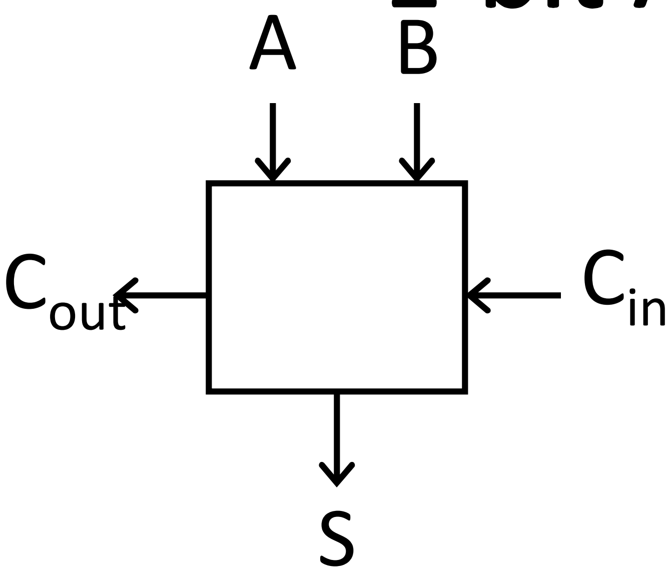
S

C_{in} \ AB	00	01	11	10
0	0	1	0	1
1	1	0	1	0

C_{out}

C_{in} \ AB	00	01	11	10
0	0	0	1	0
1	0	1	1	1

1-bit Adder with Carry



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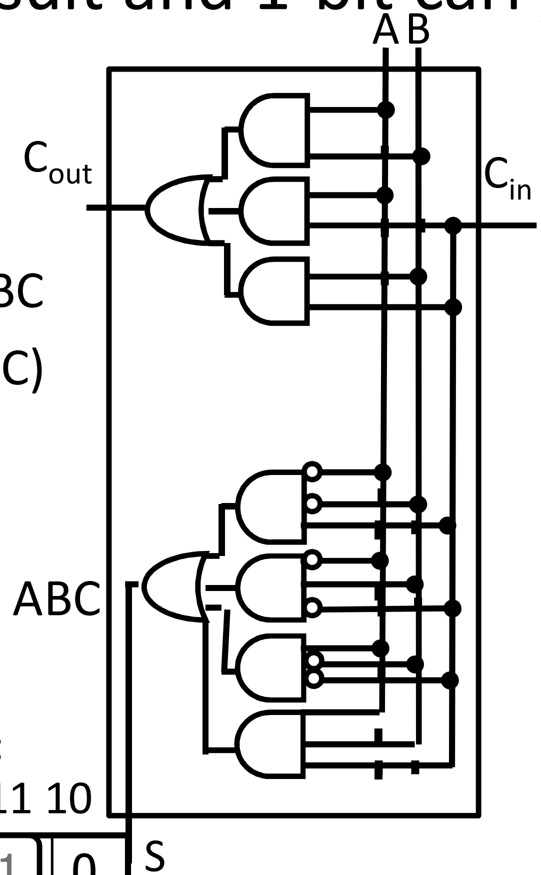


- $S = \overline{A}\overline{B}C + \overline{A}B\overline{C} + A\overline{BC} + ABC$
- $S = \overline{A}(\overline{B}C + B\overline{C}) + A(\overline{BC} + BC)$
- $S = \overline{A}(B \oplus C) + A(\overline{B \oplus C})$
- $S = A \oplus (B \oplus C)$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$
- $C_{out} = AB + AC + BC$

A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

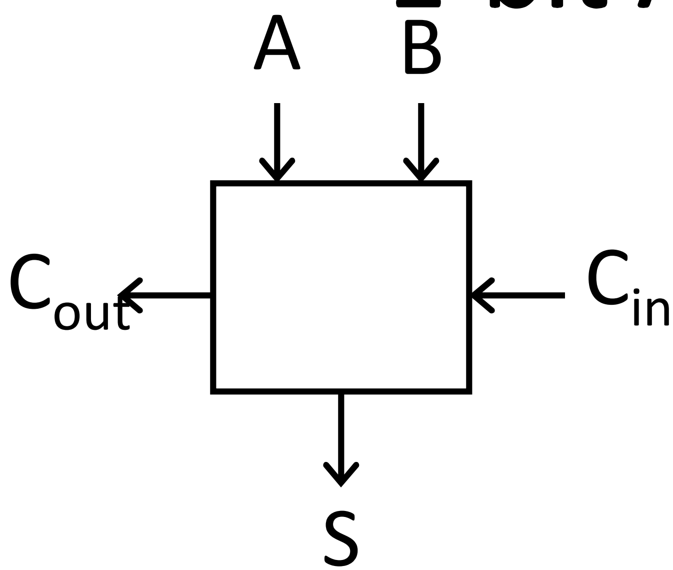
S

C_{in} \ AB	00	01	11	10
0	0	1	0	1
1	1	0	1	0

C_{out}

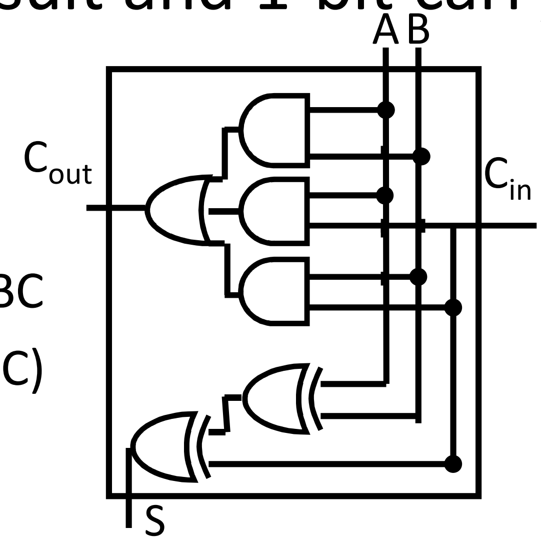
C_{in} \ AB	00	01	11	10
0	0	0	1	0
1	0	1	1	1

1-bit Adder with Carry



Full Adder

- Adds three 1-bit numbers
- Computes 1-bit result and 1-bit carry
- Can be cascaded



A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

- $S = \overline{A}\overline{B}C + \overline{A}B\overline{C} + A\overline{BC} + ABC$
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- $C_{out} = AB + AC + BC$

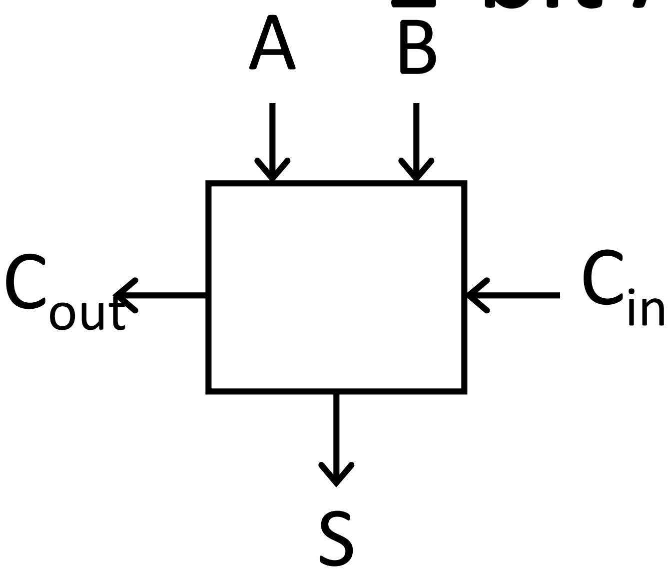
S

C_{in} \ AB	00	01	11	10
0	0	1	0	1
1	1	0	1	0

C_{out}

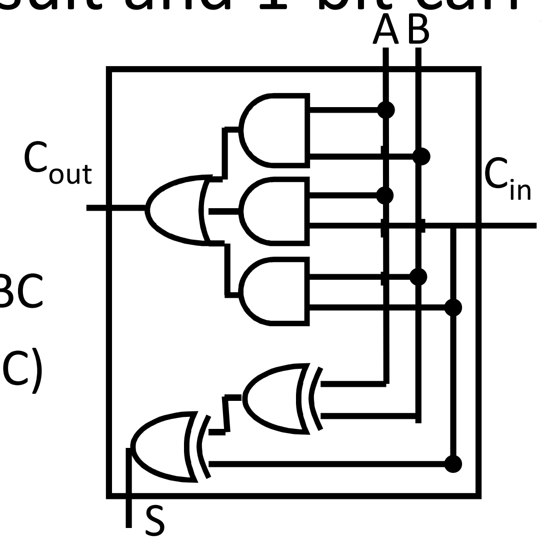
C_{in} \ AB	00	01	11	10
0	0	0	1	0
1	0	1	1	1

1-bit Adder with Carry



Full Adder

- Adds three 1-bit numbers
- Computes 1-bit result and 1-bit carry
- Can be cascaded



A	**B**	**C_{in}**	**C_{out}**	**S**
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
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1	1	1	1	1

- $S = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}\bar{C} + ABC$
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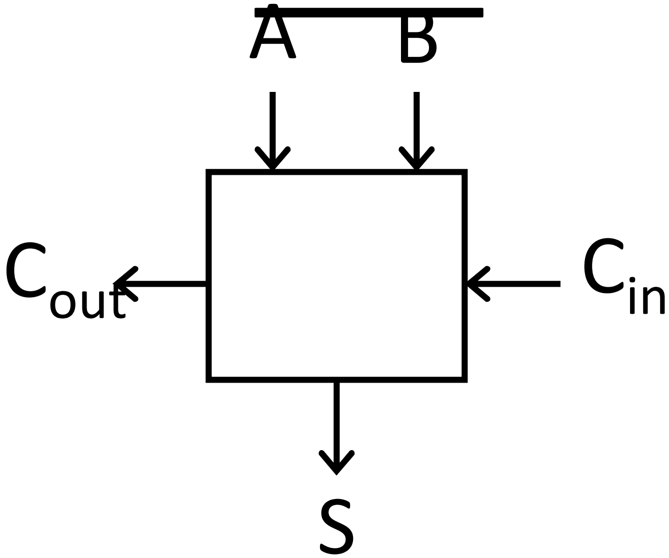
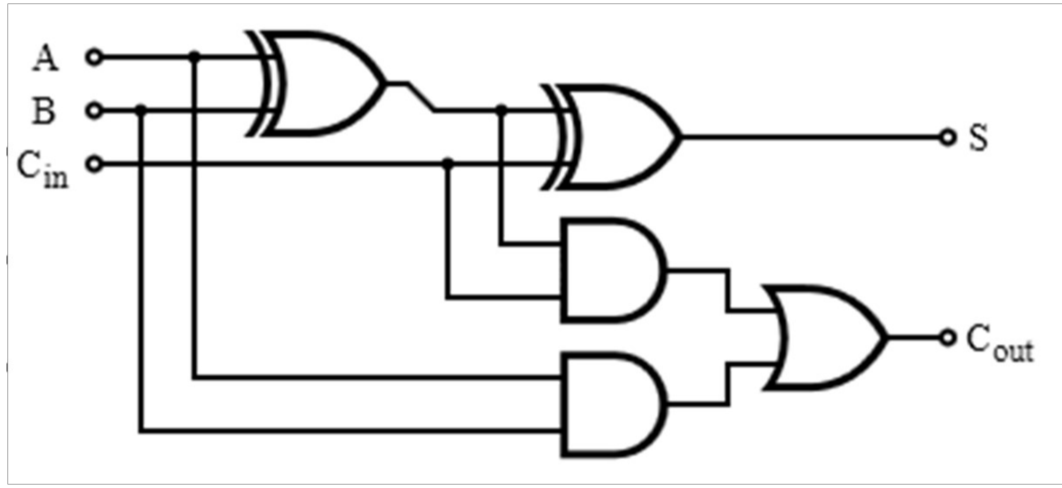
- $C_{out} = \bar{A}BC + A\bar{B}C + AB(\bar{C} + C)$
- $C_{out} = \bar{A}BC + A\bar{B}C + AB$
- $C_{out} = (\bar{A}B + A\bar{B})C + AB$
- $C_{out} = (A \oplus B)C + AB$

Lab1 1-bit Adder with Carry



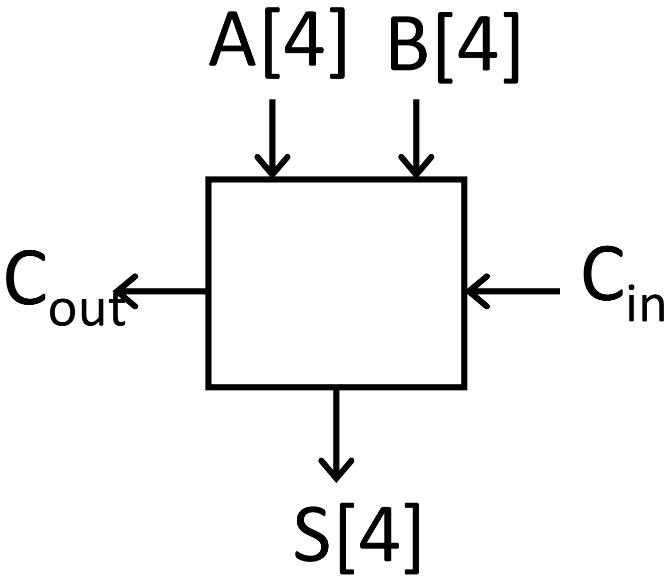


arry

A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

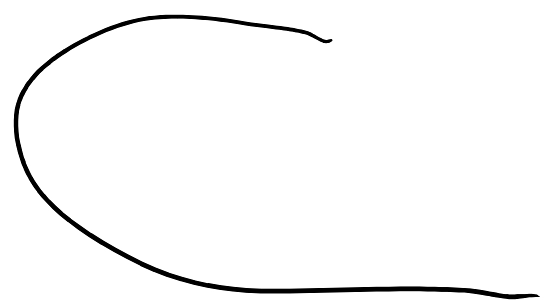
- $S = \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + ABC$
- $S = \overline{A}(\overline{B}C + B\overline{C}) + A(\overline{B}\,\overline{C} + BC)$
- $S = \overline{A}(B \oplus C) + A(\overline{B \oplus C})$
- $S = A \oplus (B \oplus C)$
- $S = A \oplus B \oplus C$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB(\overline{C} + C)$
- $C_{out} = \overline{A}BC + A\overline{B}C + AB$
- $C_{out} = (\overline{A}B + A\overline{B})C + AB$
- $C_{out} = (A \oplus B)C + AB$

4-bit Adder

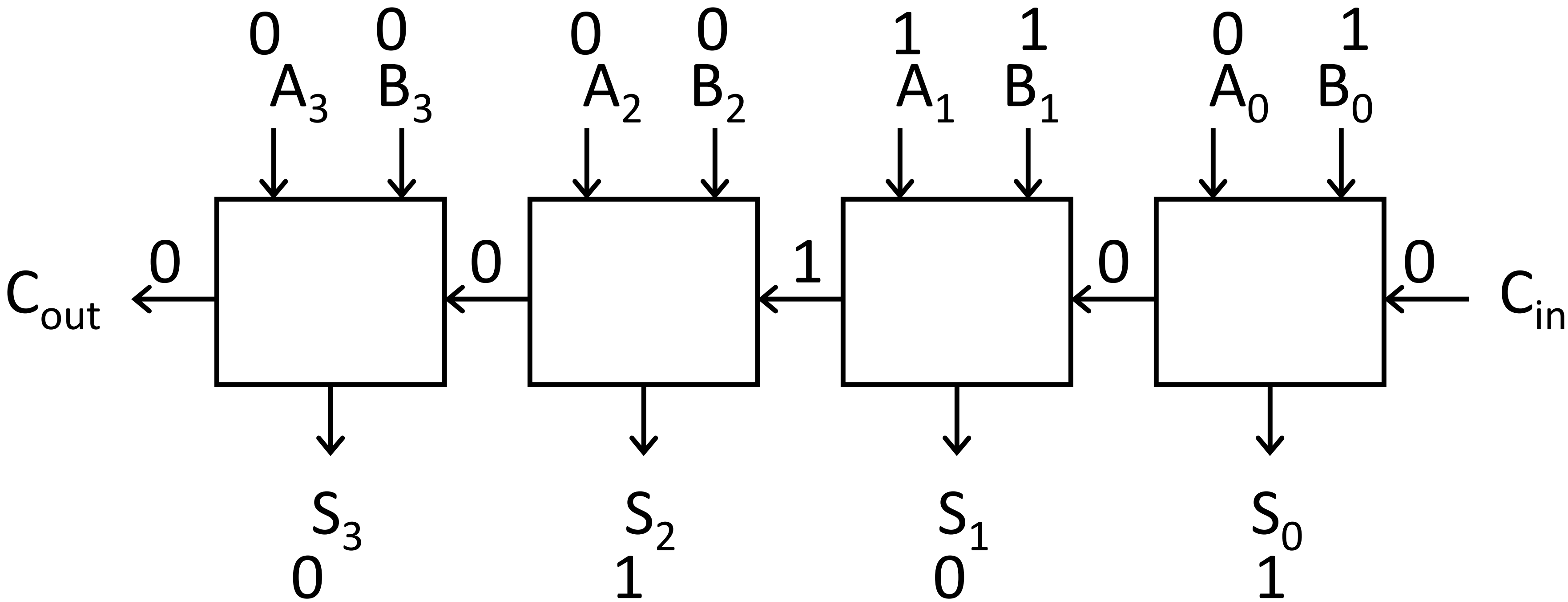


4-Bit Full Adder

- Adds two 4-bit numbers and carry in
- Computes 4-bit result and carry out
- Can be cascaded



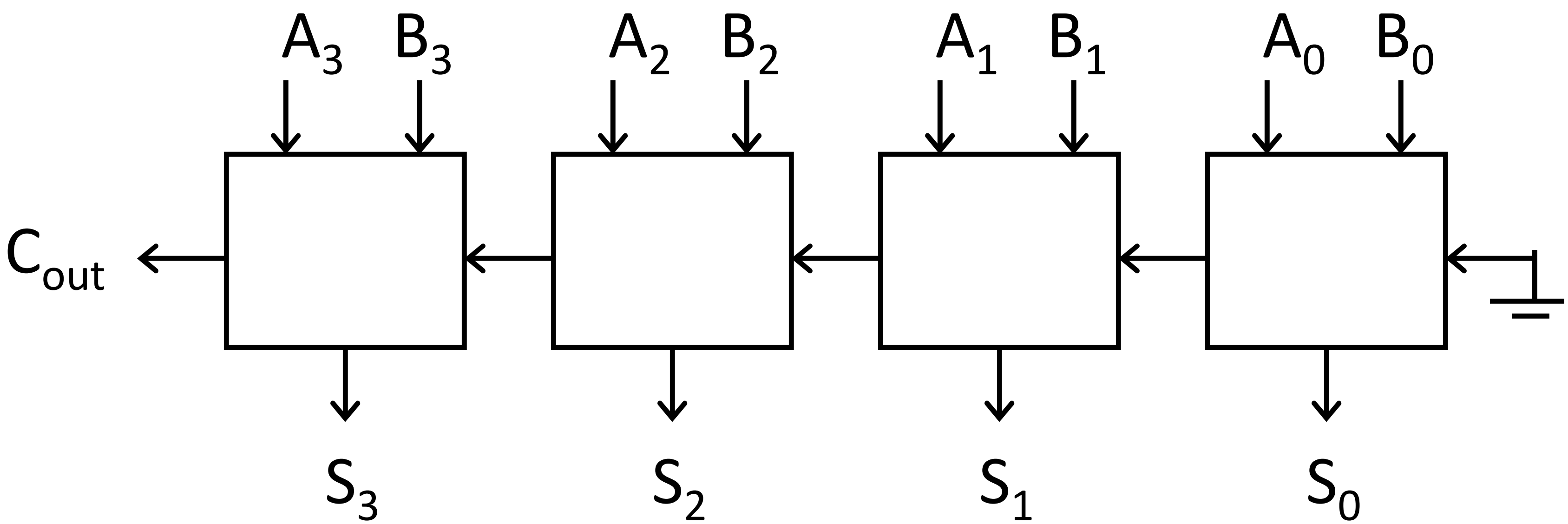
4-bit Adder



- Adds two 4-bit numbers, along with carry-in
- Computes 4-bit result and carry out

- Carry-out = overflow indicates result does not fit in 4 bits

4-bit Adder



- Adds two 4-bit numbers, along with carry-in
- Computes 4-bit result and carry out

- Carry-out = overflow indicates result does not fit in 4 bits

Takeaway

Digital computers are implemented via logic circuits and thus represent *all* numbers in binary (base 2).

We (humans) often write numbers as decimal and hexadecimal for convenience, so need to be able to convert to binary and back (to understand what computer is doing!).

Adding two 1-bit numbers generalizes to adding two numbers of any size since 1-bit full adders can be cascaded.

Today's Lecture

Binary Operations

- Number representations
- One-bit and four-bit adders
- Negative numbers and two's compliment
- Addition (two's compliment)
- Subtraction (two's compliment)

Next Goal

How do we subtract two binary numbers?

Equivalent to adding with a negative number

How do we represent negative numbers?

1st Attempt: Sign/Magnitude Representation

First Attempt: Sign/Magnitude Representation

- 1 bit for sign (0=positive, 1=negative)
- N-1 bits for magnitude

$\underline{0}111 = 7$

$\underline{1}111 = -7$

Problem?

- Two zero's: +0 different than -0

$\underline{0}000 = +0$

$\underline{1}000 = -0$

- Complicated circuits
- -2 + 1 = ???

IBM 7090, 1959:
"a second-generation transistorized version of the earlier IBM 709 vacuum tube mainframe computers"

Second Attempt: One’s complement

Second Attempt: One’s complement

- Leading 0’s for positive and 1’s for negative
- Negative numbers: complement the positive number

0111 = 7

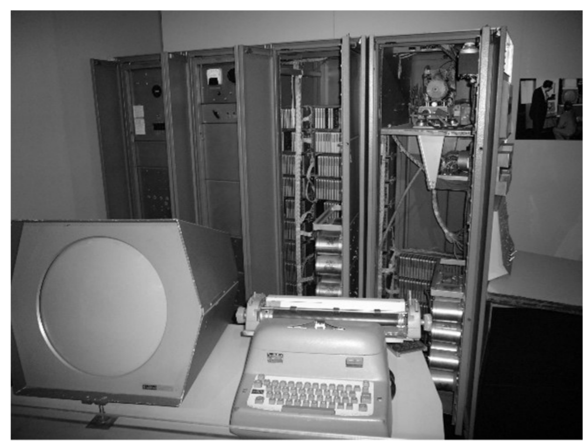
1000 = -7

Problem?

- Two zero’s still: +0 different than -0
- -1 if offset from two’s complement
- Complicated circuits
  - Carry is difficult

0000 = +0

1111 = -0



PDP 1

Two's Complement Representation

What is used: Two's Complement Representation

Nonnegative numbers are represented as usual

- 0 = 0000, 1 = 0001, 3 = 0011, 7 = 0111

Leading 1's for negative numbers

To negate any number:

- complement *all* the bits (i.e. flip all the bits)
- then add 1
- -1: 1 $\Rightarrow$ 0001 $\Rightarrow$ 1110 $\Rightarrow$ 1111
- -3: 3 $\Rightarrow$ 0011 $\Rightarrow$ 1100 $\Rightarrow$ 1101
- -7: 7 $\Rightarrow$ 0111 $\Rightarrow$ 1000 $\Rightarrow$ 1001
- -8: 8 $\Rightarrow$ 1000 $\Rightarrow$ 0111 $\Rightarrow$ 1000
- -0: 0 $\Rightarrow$ 0000 $\Rightarrow$ 1111 $\Rightarrow$ 0000 (this is good, -0 = +0)

Two's Complement Representation

Is there only one zero!

$$
\begin{array}{r}
0 = 0000 \\
\overline{0} = 1111 \\
+1 \\
\hline
0 = 0000
\end{array}
$$

One more example. How do we represent -20?

$$
\begin{array}{r}
20 = 0001\ \ 0100 \\
\overline{20} = 1110\ \ 1011 \\
+1 \\
\hline
-20 = 1110\ \ 1100
\end{array}
$$

Two’s Complement

Non-negatives (as usual):	Negatives (two’s complement) flip	then add 1
+0 = 0000		
+1 = 0001		
+2 = 0010		
+3 = 0011		
+4 = 0100		
+5 = 0101		
+6 = 0110		
+7 = 0111		
+8 = 1000		

Two's Complement

Non-negatives (as usual):	Negatives (two's complement) flip	then add 1
+0 = 0000	$\overline{0}$ = 1111	-0 = 0000
+1 = 0001	$\overline{1}$ = 1110	-1 = 1111
+2 = 0010	$\overline{2}$ = 1101	-2 = 1110
+3 = 0011	$\overline{3}$ = 1100	-3 = 1101
+4 = 0100	$\overline{4}$ = 1011	-4 = 1100
+5 = 0101	$\overline{5}$ = 1010	-5 = 1011
+6 = 0110	$\overline{6}$ = 1001	-6 = 1010
+7 = 0111	$\overline{7}$ = 1000	-7 = 1001
+8 = 1000	$\overline{8}$ = 0111	-8 = 1000

Two's Complement vs. Unsigned

4 bit
Two's
Complement
-8 … 7

-1 =	1111	= 15
-2 =	1110	= 14
-3 =	1101	= 13
-4 =	1100	= 12
-5 =	1011	= 11
-6 =	1010	= 10
-7 =	1001	= 9
-8 =	1000	= 8
+7 =	0111	= 7
+6 =	0110	= 6
+5 =	0101	= 5
+4 =	0100	= 4
+3 =	0011	= 3
+2 =	0010	= 2
+1 =	0001	= 1
0 =	0000	= 0

4 bit
Unsigned
Binary
0 … 15

Clicker Question!

What is the value of the 2s complement number 11010

a) 26
b) 6
c) -6
d) -10
e) -26

Clicker Question!

What is the value of the 2s complement number 11010

a) 26
b) 6
c) -6
d) -10
e) -26

$$\begin{array}{r} 11010 \\ \\ 00101 \text{ (flip)} \\ +1 \\ \hline -6 = 00110 \end{array}$$

Two's Complement Facts

Signed two's complement

- Negative numbers have leading 1's
- zero is unique: +0 = - 0
- wraps from largest positive to largest negative

N bits can be used to represent

- unsigned: range $0 \ldots 2^N-1$
  - eg: 8 bits $\Rightarrow$ 0…255
- signed (two's complement): $-(2^{N-1}) \ldots (2^{N-1} - 1)$
  - E.g.: 8 bits $\Rightarrow$ (1000 000) … (0111 1111)
  - -128 … 127

Sign Extension & Truncation

Extending to larger size

- 1111 = -1
- 1111 1111 = -1
- 0111 = 7
- 0000 0111 = 7

Truncate to smaller size

- 0000 1111 = 15
- BUT, ~~0000~~ 1111 = 1111 = -1

Two’s Complement Addition

Addition with two’s complement signed numbers

Addition as usual. Ignore the sign. It just works!

Examples

- 1 + -1 =
- -3 + -1 =
- -7 + 3 =
- 7 + (-3) =

-1 =	1111	= 15
-2 =	1110	= 14
-3 =	1101	= 13
-4 =	1100	= 12
-5 =	1011	= 11
-6 =	1010	= 10
-7 =	1001	= 9
-8 =	1000	= 8
+7 =	0111	= 7
+6 =	0110	= 6
+5 =	0101	= 5
+4 =	0100	= 4
+3 =	0011	= 3
+2 =	0010	= 2
+1 =	0001	= 1
0 =	0000	= 0

Two's Complement Addition

Addition with two's complement signed numbers

Addition as usual. Ignore the sign. It just works!

Examples

- 1 + -1 = 0001 + 1111 =
- -3 + -1 = 1101 + 1111 =
- -7 + 3 = 1001 + 0011 =
- 7 + (-3) = 0111 + 1101 =

-1 =	1111	= 15
-2 =	1110	= 14
-3 =	1101	= 13
-4 =	1100	= 12
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-8 =	1000	= 8
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+6 =	0110	= 6
+5 =	0101	= 5
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+3 =	0011	= 3
+2 =	0010	= 2
+1 =	0001	= 1
0 =	0000	= 0

Two’s Complement Addition

Addition with two’s complement signed numbers

Addition as usual. Ignore the sign. It just works!

Examples

- 1 + -1 = 0001 + 1111 = 0000 (0)
- -3 + -1 = 1101 + 1111 = 1100 (-4)
- -7 + 3 = 1001 + 0011 = 1100 (-4)
- 7 + (-3) = 0111 + 1101 = 0100 (4)

-1 =	1111	= 15
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-3 =	1101	= 13
-4 =	1100	= 12
-5 =	1011	= 11
-6 =	1010	= 10
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-8 =	1000	= 8
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+6 =	0110	= 6
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+2 =	0010	= 2
+1 =	0001	= 1
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-4 =	1100	= 12
-5 =	1011	= 11
-6 =	1010	= 10
-7 =	1001	= 9
-8 =	1000	= 8
+7 =	0111	= 7
+6 =	0110	= 6
+5 =	0101	= 5
+4 =	0100	= 4
+3 =	0011	= 3
+2 =	0010	= 2
+1 =	0001	= 1
0 =	0000	= 0

Which of the following has problems?

a) 7 + 1
b) -7 + -3
c) -7 + -1
d) Only (a) and (b) have problems
e) They all have problems

Clicker Question

Two’s Complement Addition

Addition with two’s complement signed numbers

Addition as usual. Ignore the sign. It just works!

Examples

- 1 + -1 = 0001 + 1111 = 0000 (0)
- -3 + -1 = 1101 + 1111 = 1100 (-4)
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-3 =	1101	= 13
-4 =	1100	= 12
-5 =	1011	= 11
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-7 =	1001	= 9
-8 =	1000	= 8
+7 =	0111	= 7
+6 =	0110	= 6
+5 =	0101	= 5
+4 =	0100	= 4
+3 =	0011	= 3
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+1 =	0001	= 1
0 =	0000	= 0

Which of the following has problems?

a) 7 + 1
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c) -7 + -1
d) Only (a) and (b) have problems
e) They all have problems

Clicker Question

Two’s Complement Addition

Addition with two’s complement signed numbers

Addition as usual. Ignore the sign. It just works!

Examples

- 1 + -1 = 0001 + 1111 = 0000 (0)
- -3 + -1 = 1101 + 1111 = 1100 (-4)
- -7 + 3 = 1001 + 0011 = 1100 (-4)
- 7 + (-3) = 0111 + 1101 = 0100 (4)

Which of the following has problems?

a) 7 + 1 = 1000 overflow

b) -7 + -3 = 1 0110 overflow

c) -7 + -1 = 1000 fine

d) Only (a) and (b) have problems

e) They all have problems

Clicker Question

-1 =	1111	= 15
-2 =	1110	= 14
-3 =	1101	= 13
-4 =	1100	= 12
-5 =	1011	= 11
-6 =	1010	= 10
-7 =	1001	= 9
-8 =	1000	= 8
+7 =	0111	= 7
+6 =	0110	= 6
+5 =	0101	= 5
+4 =	0100	= 4
+3 =	0011	= 3
+2 =	0010	= 2
+1 =	0001	= 1
0 =	0000	= 0

Next Goal

In general, how do we detect and handle overflow?

Overflow

When can overflow occur?

- adding a negative and a positive?
- adding two positives?
- adding two negatives?

Overflow

When can overflow occur?

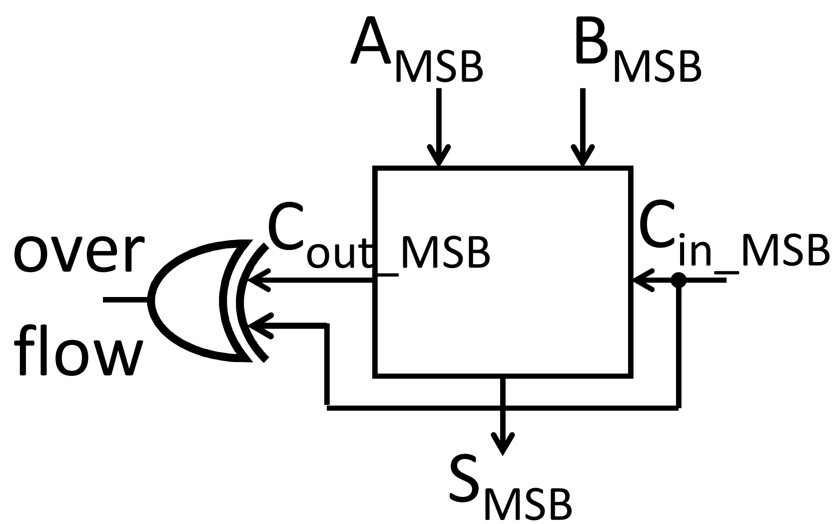
- adding a negative and a positive?
  - Overflow *cannot occur* (Why?)
  - Always subtract larger magnitude from smaller
- adding two positives?
  - Overflow *can occur* (Why?)
  - Precision: Add two positives, and get a negative number!
- adding two negatives?
  - Overflow *can occur* (Why?)
  - Precision: add two negatives, get a positive number!

Rule of thumb:

- Overflow happens iff
  carry into msb != carry out of msb

Overflow

When can overflow occur?



MSB

A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Wrong Sign

Wrong Sign

Rule of thumb:

- Overflow happens iff
  carry into msb != carry out of msb

YouTube

Shared publicly - Dec 1, 2014

We never thought a video would be watched in numbers greater than a 32-bit integer (=2,147,483,647 views), but that was before we met PSY. "Gangnam Style" has been viewed so many times we had to upgrade to a 64-bit integer (9,223,372,036,854,775,808)!

Hover over the counter in PSY's video to see a little math magic and stay tuned for bigger and bigger numbers on YouTube.

Today's Lecture

Binary Operations

- Number representations
- One-bit and four-bit adders
- Negative numbers and two's compliment
- Addition (two's compliment)
- Detecting and handling overflow
- Subtraction (two's compliment)

Binary Subtraction

Why create a new circuit?

Just use addition using two’s complement math

- How?

Binary Subtraction

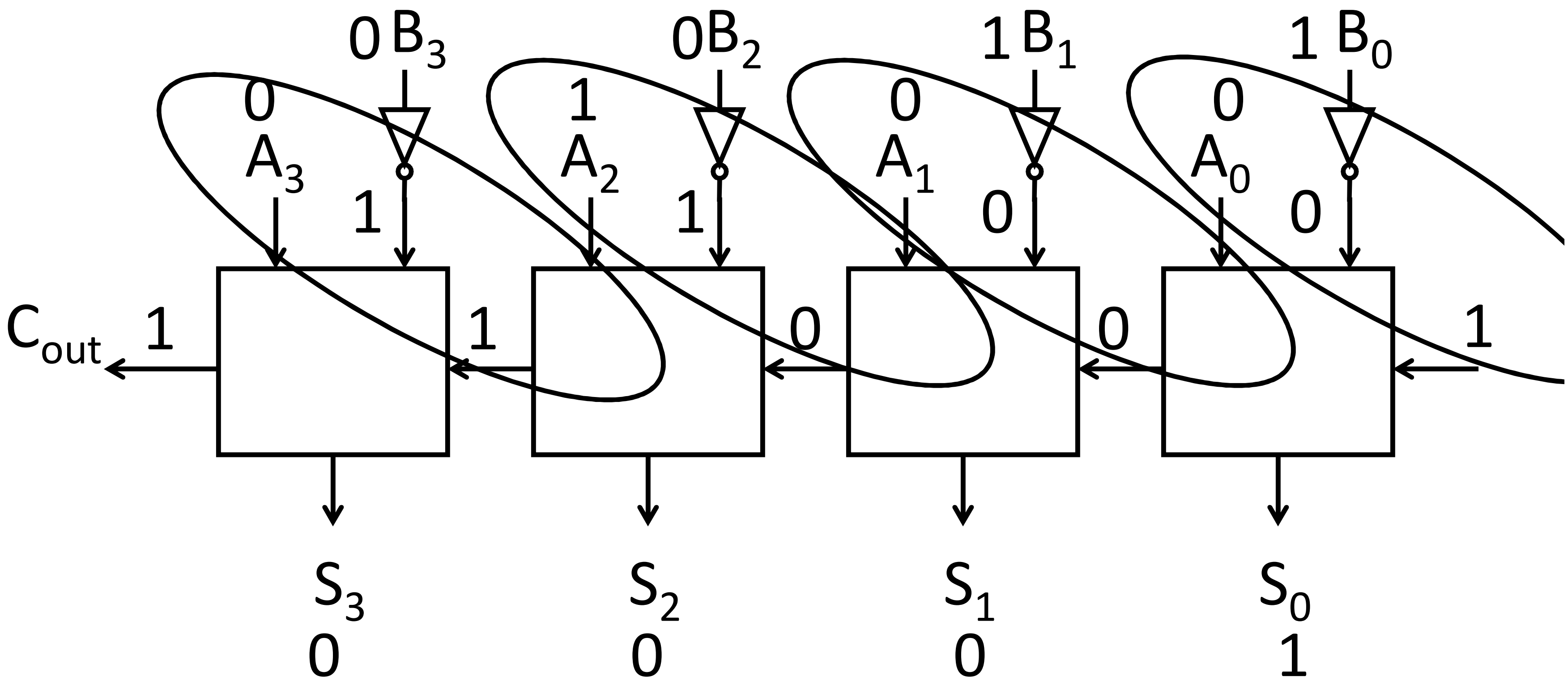
Two's Complement Subtraction

- Subtraction is simply addition,

  where one of the operands has been negated

  - Negation is done by inverting all bits and adding one

    $A - B = A + (-B) = A + (\overline{B} + 1)$



Binary Subtraction

Two's Complement Subtraction

- Subtraction is simply addition,

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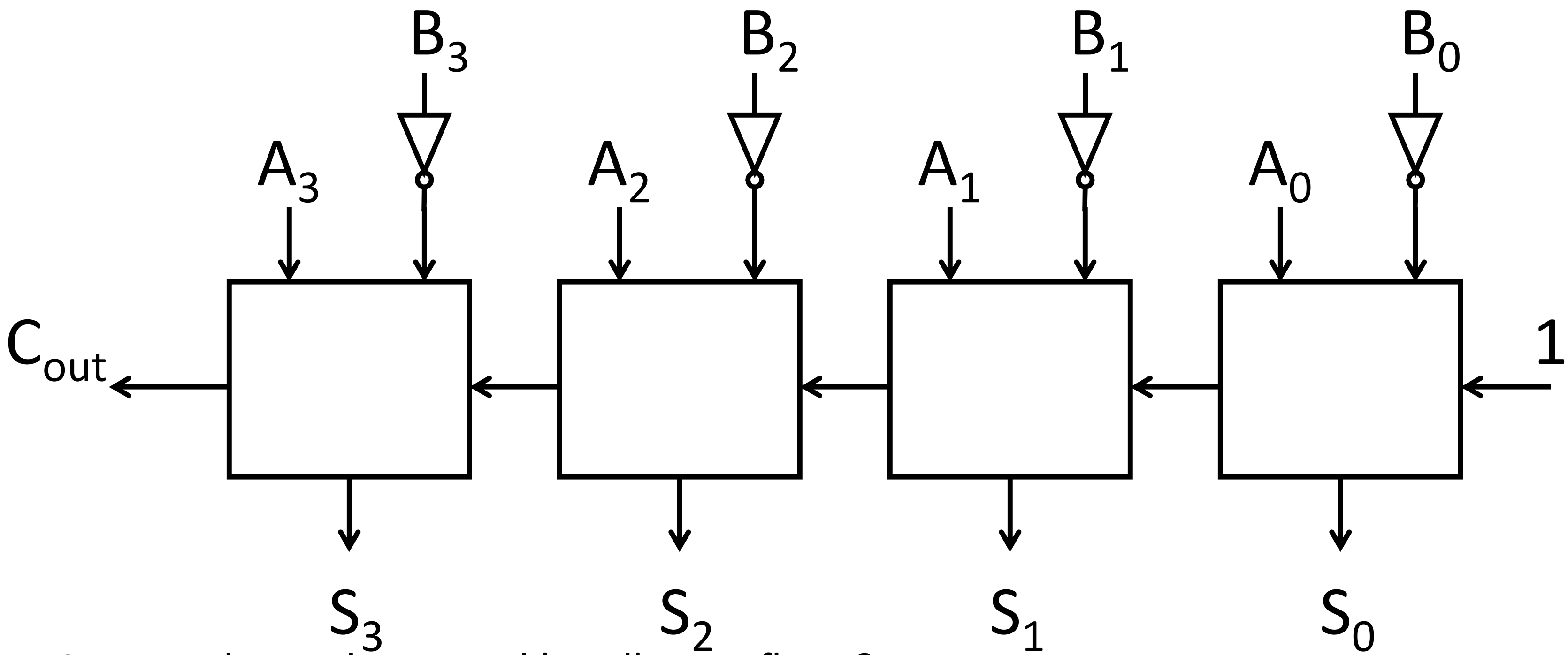
  $A - B = A + (-B) = A + (\overline{B} + 1)$

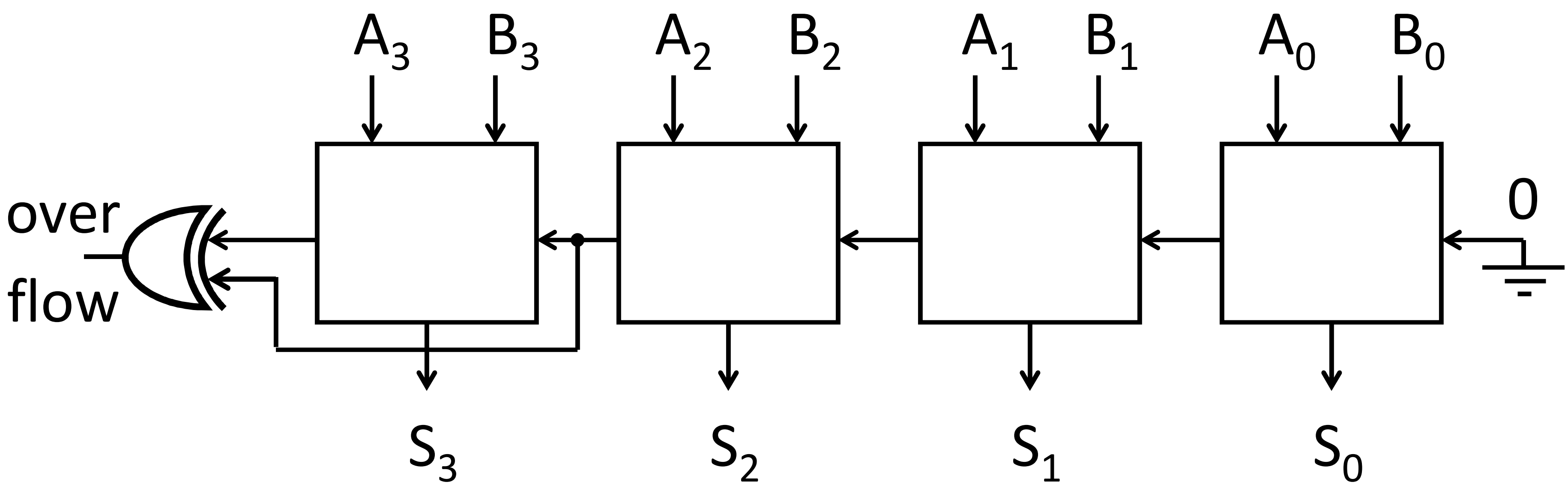


Q: How do we detect and handle overflows?

Q: What if (-B) overflows?

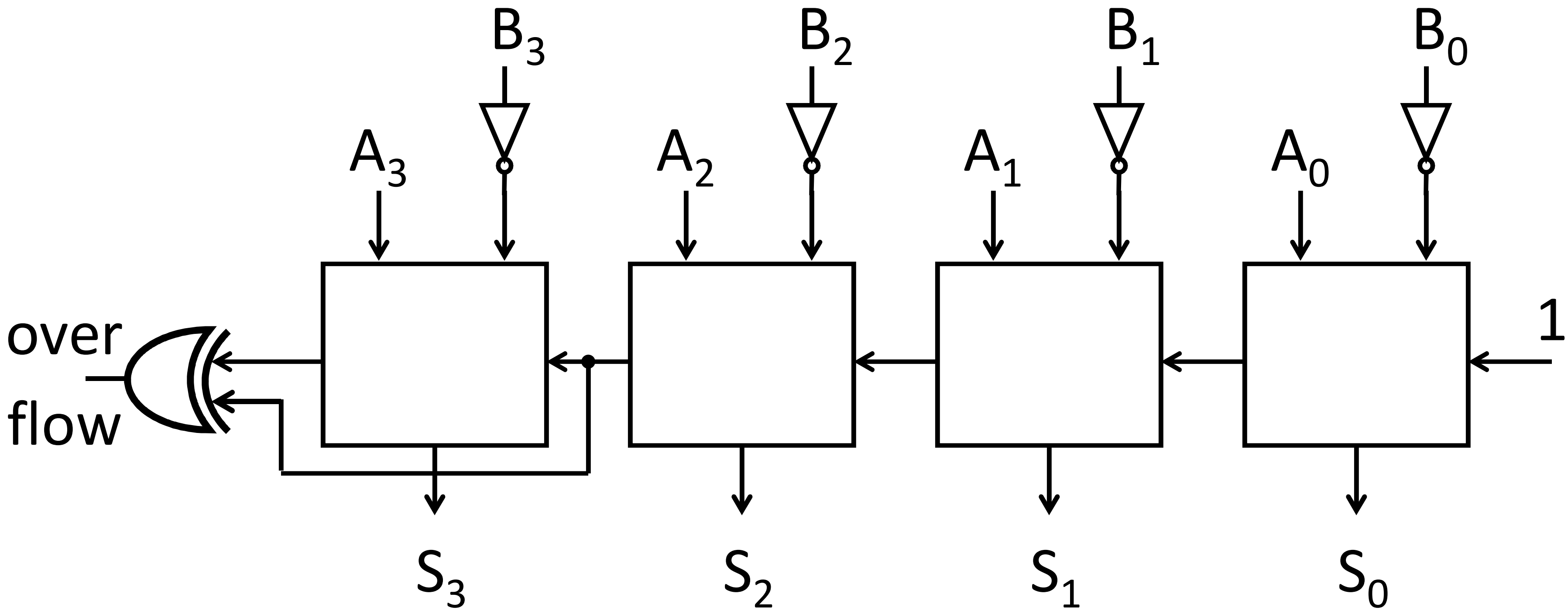
Two's Complement Adder

Two's Complement Adder with overflow detection



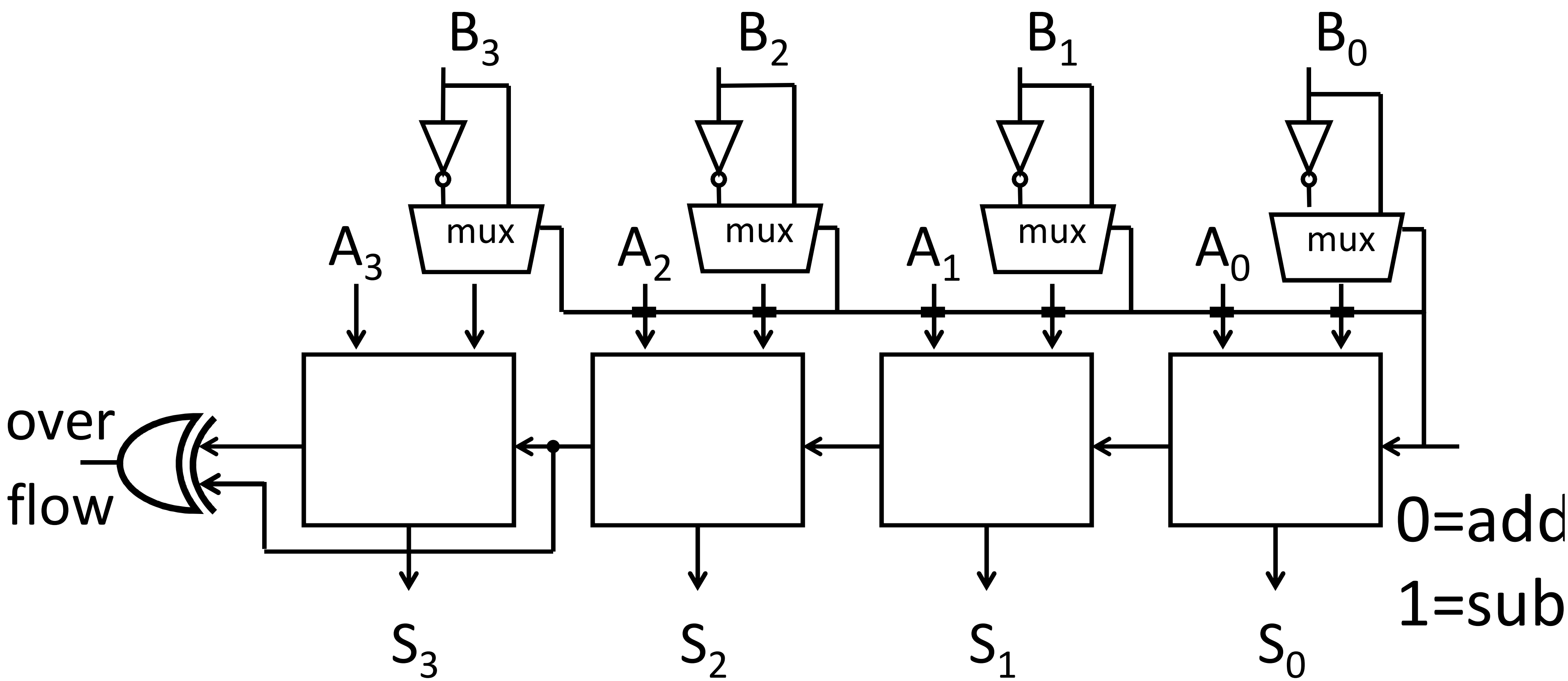
Two's Complement Adder

Two's Complement Subtraction with overflow detection



Two’s Complement Adder

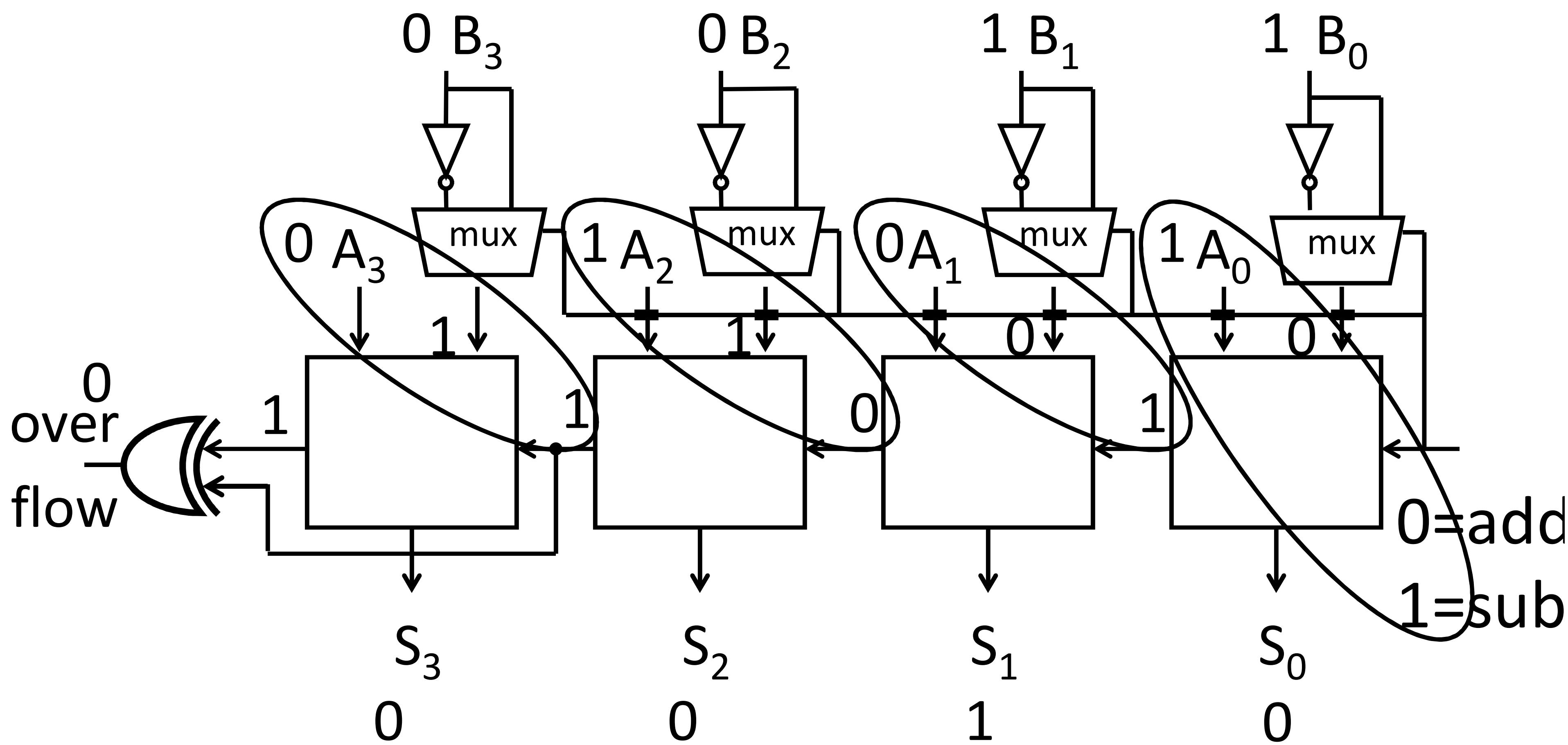
Two’s Complement Adder with overflow detection



Note: 4-bit adder is drawn for illustrative purposes and may not represent the optimal design.

Two's Complement Adder

Two's Complement Adder with overflow detection



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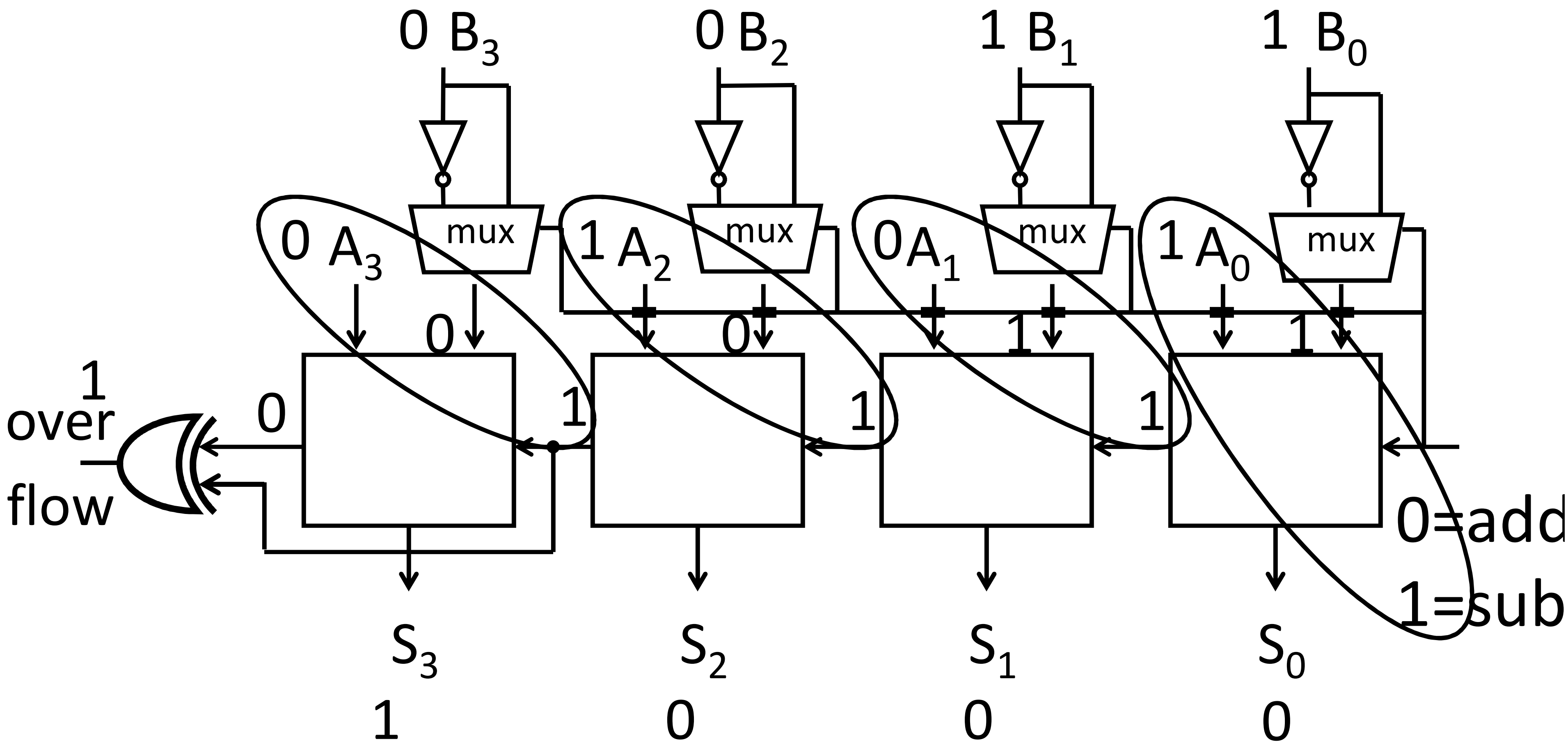
Two's Complement Adder

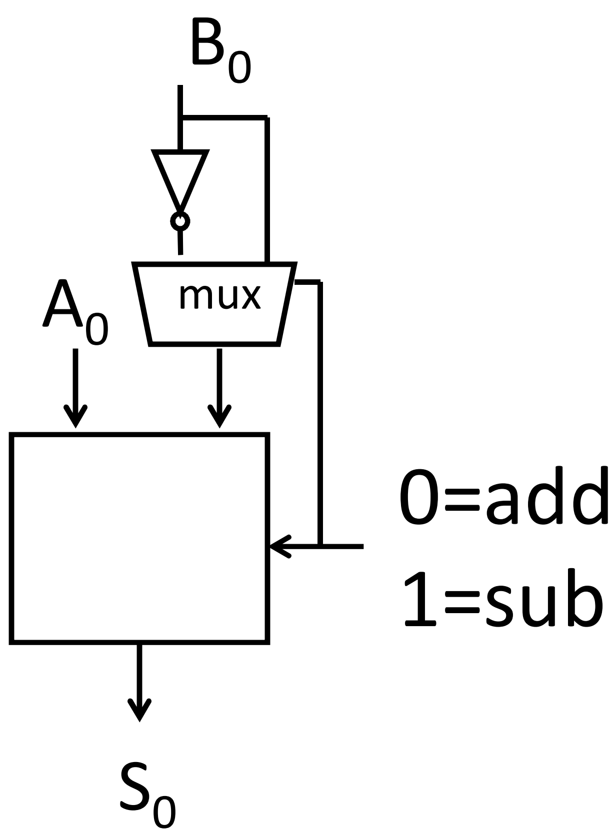
Two's Complement Adder with overflow detection

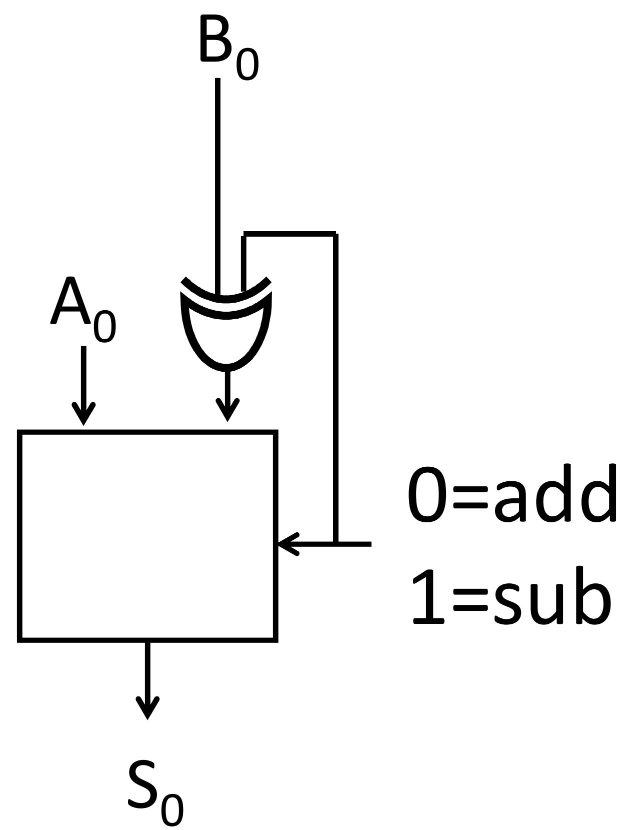


Note: 4-bit adder is drawn for illustrative purposes and may not represent the optimal design.

Put it together *better*

Two's Complement Adder with overflow detection





Before: 2 inverters, 2 AND gates, 1 OR gate

After: 1 XOR gate

Takeaways

Digital computers are implemented via logic circuits and thus represent *all* numbers in binary (base 2).

We write numbers as decimal or hex for convenience and need to be able to convert to binary and back (to understand what the computer is doing!).

Adding two 1-bit numbers generalizes to adding two numbers of any size since 1-bit full adders can be cascaded.

Using Two's complement number representation simplifies adder Logic circuit design (0 is unique, easy to negate). Subtraction is adding, where one operand is negated (two's complement; to negate: flip the bits and add 1).

Overflow if sign of operands A and B != sign of result S.

Can detect overflow by testing C_{in} != C_{out} of the most significant bit (msb), which only occurs when previous statement is true.

Summary

We can now implement combinational logic circuits

- Design each block
  - Binary encoded numbers for compactness
- Decompose large circuit into manageable blocks
  - 1-bit Half Adders, 1-bit Full Adders,
    n-bit Adders via cascaded 1-bit Full Adders, ...
- Can implement circuits using NAND or NOR gates
- Can implement gates using use PMOS and NMOS-transistors
- And can add and subtract numbers (in two's compliment)!
- Next time, state and finite state machines…