Homework 4 Solutions

CS322: Summer 2004

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1. (a) Here is the GEPP applied to the given matrix by hand:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}
= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -1 & 4 \end{pmatrix}
$$

$$\vdots$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}.
$$

It is observed that the last column of U increases exponentially while all of the other diagonal entries are 1 and the rest are 0. This shows ill-conditioning as when n is large, the norm of U is much larger than the norm of the original matrix while the norm of L is about the same as that of A (Recall that if if $A = LU$ and $\|L\|\|U\|$ is much greater than $\|A\|$, then A is ill-conditioned).

(b) See other files for MATLAB script and plots. Error, residual, and condition number are measured in 1-norm. Your graphs may look different if you use other norms and/or different RHS. The error grows in less than linear while the norm of the residual and condition number grow exponentially.

2. The given linear system is equivalent to

$$\begin{aligned} U_1 x &= b && (1) \\ Bx + U_2 y &= c && (2) \end{aligned}$$

Note that (2) is equivalent to

$$U_2 y = c - Bx \qquad (3)$$

Hence, we can compute x by performing back-substitution on (1). After x is known, we can compute the right-hand-side of (3). Therefore, we can solve for y by performing another back-substitution on (3).

Recall that U_1 is an $n \times n$ matrix and U_2 is $m \times m$. First back-substitution on U_1 to solve for x requires $n^2 + O(n)$ flops. Multiply Bx requires $2mn + O(m+n)$ flops. Subtract Bx from c requires n flops. Second back-substitution on U_2 requires $m^2 + O(m)$ flops. Total flops is therefore $(m+n)^2 + O(m+n)$.

Straightforwardly solve the linear system would require $\frac{2}{3}(n+m)^3 + O((n+m)^2)$ flops.

3. Manipulate the given equation as follow

$$\begin{aligned} x &= B^{-1}(2A+I)(C^{-1}+A)b \\ Bx &= (2A+I)C^{-1}b + (2A+I)Ab && (4) \end{aligned}$$

To compute $y \equiv C^{-1}b$, solve $Cy = b$ for y. The right-hand-side of (4) is now known so can use GEPP to solve for x.

4. (a) The given matrix is orthogonal means

$$\begin{aligned} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^T \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} &= \begin{pmatrix} A^T & 0 \\ B^T & C^T \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \\ &= I \\ &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. && (5) \end{aligned}$$

From (5), we see that $A^TA = I$, $A^TB = 0$, and $B^TB + C^TC = I$. The first one, $A^TA = I$ implies that A is orthogonal and nonzero. Also, $A^TB = 0$, so $B = 0$ since $A \neq 0$ from previously. Lastly, $B^TB + C^TC = C^TC = I$ since $B = 0$ so C is orthogonal, too.

(b)

$$\begin{aligned} H^T &= \left(I - 2\frac{vv^T}{v^Tv}\right)^T \\ &= I^T - 2\frac{(vv^T)^T}{(v^Tv)^T} \\ &= I - 2\frac{vv^T}{v^Tv} \\ &= H \end{aligned}$$

Therefore, H is symmetric. Its orthogonality is shown below

$$\begin{aligned} H^TH &= \left(I - 2\frac{vv^T}{v^Tv}\right)\left(I - 2\frac{vv^T}{v^Tv}\right) \\ &= I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^Tvv^T}{v^Tvv^Tv} \\ &= I - 4\frac{vv^T}{v^Tv} + 4\frac{v(v^Tv)v^T}{(v^Tv)v^Tv} \\ &= I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv} \\ &= I \end{aligned}$$

(c) The key idea is to note that $a^Ta = \|a\|_2^2 = \alpha^2$ and $e_1^Te_1 = 1$. Then, we can proceed to simplify the expression

$$\begin{aligned} Ha &= \left(I - 2\frac{(a - \alpha e_1)(a^T - \alpha e_1^T)}{(a^T - \alpha e_1^T)(a - \alpha e_1)}\right)a \\ &= \left(I - 2\frac{(a - \alpha e_1)(a^T - \alpha e_1^T)}{a^Ta - 2\alpha e_1^Ta + \alpha^2 e_1^Te_1}\right)a \\ &= \left(I - 2\frac{(a - \alpha e_1)(a^T - \alpha e_1^T)}{\alpha^2 - 2\alpha e_1^Ta + \alpha^2}\right)a \\ &= \left(I - \frac{(a - \alpha e_1)(a^T - \alpha e_1^T)}{\alpha^2 - \alpha e_1^Ta}\right)a \end{aligned}$$

$$
\begin{aligned}
&= a - \frac{(a - \alpha e_1)(a^T - \alpha e_1^T)}{\alpha^2 - \alpha e_1^T a} a \\
&= a - \frac{(a - \alpha e_1)(a^T a - \alpha e_1^T a)}{\alpha^2 - \alpha e_1^T a} \\
&= a - \frac{(a - \alpha e_1)(\alpha^2 - \alpha e_1^T a)}{\alpha^2 - \alpha e_1^T a} \\
&= a - a + \alpha e_1 \\
&= \alpha e_1.
\end{aligned}
$$