Problem set 6

Due Friday, August 6th at 1:00 pm

Question 1 (40 points) Consider the nonlinear least squares problem of finding the best function of the form

$$f(x) = p_1 e^{p_2 x}$$

to fit the data points $\{(0,2),(1,0.7),(2,0.3),(3,0.1)\}$. Let $F(p) : \Re^2 \to \Re^4$ be the residual function, i.e. $F_i(p) = p_1 e^{p_2 x_i} - y_i$. What is the Jacobian of F? Write a matlab script to solve this nonlinear least squares problem using the Gauss-Newton method with $p^0 = [1,0]^T$ as a starting guess (you can stop after six Gauss-Newton iterations). Use a QR factorization method (you can use the function qr in Matlab or one of the Givens rotation methods from your text) to solve the linear least squares subproblem (step 2 from your 7/30 lecture notes or in Van Loan). Make a table showing the first six iterates $p^0, ...p^5$ and the residual, $\|F(p^i)\|_2^2$ at each iteration. Plot the exponential model and the data points in a single matlab plot. Look at examples in Van Loan to see how to make the four data points show up as circles (or some other noticeable shape) that are *not* connected by lines. Experiment with other starting guesses p^0. Do you always get convergence? Do you always get convergence to the same solution? Hand in your plots and analysis and printouts of your matlab code. Email a copy of your matlab script to gunsri@cs.cornell.edu

Question 2 (20 points) Consider the function $f : \Re^2 \to \Re$ defined by

$$f(x) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2$$

At what point does f attain a minimum? Perform (by hand) one iteration of Newton's method for minimizing f using $x_0 = [2,2]^T$ as a starting point. Think about (**but do not submit answers to**) the following two questions: In what sense is this a good step? In what sense is this a bad step? Show all of your work.

Question 3 (40 points) Consider a light body orbiting a heavy body located at the origin lying in a plane. The equations of motion of the light body are given by

$$\frac{d^2x}{dt^2} = \frac{-x}{(x^2+y^2)^{1.5}}$$
$$\frac{d^2y}{dt^2} = \frac{-y}{(x^2+y^2)^{1.5}}$$

Write the equations of motion as an equivalent system of four first order differential equations. Write two matlab scripts to solve for the trajectory of the orbiting body if it starts at (1,1) at time zero with velocity (0,1). In one matlab script, solve the IVP using the order 1 Adams-Bashforth method (p.351) and in the other script solve the IVP using the order 2 Adams-Bashforth method. In both cases use a step size $h = .005$ and take about 10,000 steps. Plot the resulting trajectories on two separate plots. What do you observe?

The orbiting body should satisfy the conservation of energy condition $dE/dt = 0$ where

$$E(x, y, V_x, V_y) = \frac{V_x^2 + V_y^2}{2} - \frac{1}{(x^2 + y^2)^{0.5}}$$

and (V_x, V_y) is the velocity of the orbiting body. Determine computationally how well the AB1 and AB2 methods conserve energy by plotting $E(x(t), y(t), V_x(t), V_y(t)) - E_0$ as a function of t for both methods (where E_0 is the energy at time zero). What do you observe? Turn in your work, conclusions, and print outs of your matlab files. Email a copy of your matlab files to gunsri@cs.cornell.edu

Challenge (not part of the assignment) Write down the equations of motion of three bodies (planets) lying in a plane which each have some initial location and velocity and move under the influence of the gravitational pull from the other two bodies (i.e. write F=ma for the three bodies and 'solve' for a.) Write an equivalent system of first order differential equations. Solve this system using one of the methods from the text (AB2? RK4?) for some initial position and velocity data you make up. This is the "three-body problem" which does not have a closed-form solution (unlike the two body problem). You can probably figure out a way to make a "movie" out of your trajectory data using the matlab graphics functions.