**Problem set 5**

Due Friday, July 30 at 1:00 pm

**Question 1** (30 points) Sparse QR factorization. If $A \in \Re^{m \times n}$ is *almost* upper triangular (i.e. there are not very many non-zero sub-diagonal elements) Givens rotations can be selectively applied to zero the sub-diagonal entries. Suppose vectors $r$ and $p$ store the row and column of the non-zero sub-diagonal entries of $A$. In particular, entry $(r(i), p(i))$ of $A$ is nonzero for $i = 1 : k$ where $k$ is the length of $r$ and $p$. Write a matlab function SparseQR(A,r,p) that returns the upper triangular matrix $R$ in the QR factorization of $A$ and vectors $c, s$ in $\Re^k$ where $(c(i), s(i))$ are the $c$ and $s$ used in the Givens rotation zeroing $A(r(i), c(i))$. In this way the vectors $c$ and $s$ represent the orthogonal matrix $Q$ in the $QR$ factorization. You may assume that input vector $p$ satisfies $p(1) \leq p(2) \leq ... \leq p(k)$. **Efficiency matters.** (Note that $A$ could be more efficiently stored, but you may assume it is stored as a full $m \times n$ matrix.) Hand in a print out of your matlab code and email a copy to gunsri@cs.cornell.edu

**Question 2** (20 points) Consider the vectors $c$ and $s$ representing the Givens rotations applied to matrix $A$ in question 1. Write a matlab function SparseQMult(c,s,r,p,b) that returns $Qb$, where $Q$ is the orthogonal matrix represented by $c, s$. *Make sure to return $Qb$ and not $Q^T b$.* **Efficiency matters.** Hand in a print out of your matlab code and email a copy to gunsri@cs.cornell.edu

**Question 3** (25 points) Let $a \in \Re^n$ be considered as an $n \times 1$ matrix. Write out the $QR$ factorization of $a$ showing the matrix $R$ explicitly and either give a formula for the matrix $Q$ or show each element explicitly (you can do this factorization in one step if you use a Householder transformation.) What is the solution to the linear least squares problem $ax \approx b$ where $b$ is a given $n$-vector? Show that the columns of $Q$ are an orthonormal basis for $\Re^n$ and that $\frac{a}{\|a\|_2}$ is one of the basis vectors. This is an efficient way to build an orthonormal basis for $\Re^n$ from an initial basis vector $a$.

**Question 4** (10 points) If $A \in \Re^{n \times n}$ is positive definite and $X \in \Re^{n \times k}$ has rank $k$, prove that $X^T AX$ is positive definite.

**Question 5** (15 points) If $A \in \Re^{n \times n}$ is symmetric positive definite, prove that $|A(i,j)| \leq (A(i,i) + A(j,j))/2$. Hint: first establish the two inequalities $A(i,i) - 2A(i,j) + A(j,j) \geq 0$ and $A(i,i) + 2A(i,j) + A(j,j) \geq 0$.