Slide 6:

1. The car is driving at a constant speed. In that case it is equally likely to find it anywhere in the circle. If the radius of the circle is R the total length of the circle is 2*pi*R. The number of radians of the circle is 2*pi. Counting the distance in radians the pdf is 1/(2*pi). The probability is Integral from 1 to 2 of (1/(2*pi) d(rad) = (2-1)/(2*pi) = 1/(2*pi)
2. The probability of a single point on the continuous line is zero. I.e. the probability of observing 0.75 exactly when sampling from a uniform continuous distribution between 0 and 1 is exactly zero.
3. If the protein folds at a constant rate then the number of molecules dN that fold at an interval time dt is given by -dN = N*alpha*dt , where N is the number of molecules that did not fold and dN is therefore the loss of unfolded molecules to folded molecules -> -dN/N = alpha*dt -> -d(log(N)) = alpha*dt ->
 N = N0*exp(-alpha*t). N is the number of molecules that did not fold, to find out the number of molecules that fold we subtract it from the total of N0 N0(1-exp(-alpha*t)). The probability that all N0 molecules will fold after time t is therefore 1-exp(-alpha*t)

Slide 9:

1. If we use the length of the arc, then the probability density is given by $f(x) = \frac{1}{2\pi a}$, if we use radians, then the probability density is $f(\theta) = \frac{1}{2\pi}$ -- the probability density depends on the presentation of the space

2, We consider the probability density $f(x)$ which is non-zero in the interval [0,1]. $f(x) = \frac{2}{3}x^{-1/3}$. Close to zero the probability density function is approaching a value of infinite. Does it violate what we know about probabilities? No! since the probability must be defined on an interval (even very small) and the above probability density is integrable. For example, consider a small interval ε near zero. The probability (not the probability density!) of finding a sample at the interval $[0,\varepsilon]$ is $\int_0^{\varepsilon} \frac{2}{3}x^{-1/3} = x^{2/3}\Big|_0^{\varepsilon} = \varepsilon^{2/3}$ which is obviously a finite number. Is a probability density $1/x$ allowed in the same interval?

Slide 10. Normalizing the exponential distribution

$$f(x) = c\left[f'(x) = \begin{cases} \exp(-ax) & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases} \right]$$

$$\int_0^{\infty} c f'(x)\, dx = c\int_0^{\infty} \exp(-ax)\, dx = c\left[\frac{\exp(-ax)}{-a}\right]_0^{\infty} = \frac{c}{a} = 1$$

$$c = a$$

Slide 11: Normalizing the normal distribution –
Consider the probability density functions of two continuous random variables X,Y. One of the p.d.f is $f(X)$ and the second is $f(Y)$. Assume further that they are identical and are analytically given by $f(z)=\left[\left(\frac{a}{\pi}\right)^{1/2}\right]\exp\left(-az^2\right) \qquad Z=x-x_0,y-x_0$. Both of these distributions are expected to be normalized i.e. $\int_{-\infty}^{\infty} f(z)dz=1$. We now write the obvious $\int_{-\infty}^{\infty} f(x)\cdot dx\int_{-\infty}^{\infty} f(y)\cdot dy=1$ which can also be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\cdot dx\cdot dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{a}{\pi}\right)\exp\left[-a\left(x^2+y^2\right)\right]\cdot dx\cdot dy\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f}(x,y)\cdot dx\cdot dy$$

(note that we have an example here of a p.d.f of more than one variable) This is an integral on an infinite two dimensional plan. Rather than using the Cartesian coordinates x,y one can use polar coordinates to do the same integral. (this coordinate transformation are very useful in transforming random variables from one p.d.f to another. We define $r=\sqrt{x^2+y^2}$ and $\theta=\arctan(y/x)$ to re-write the integral as a sum over circular bands $(2\pi\cdot r\cdot dr)$ instead of squares $(dx\cdot dy)$. We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{a}{\pi}\right)\exp\left[-a\left(x^2+y^2\right)\right]\cdot dx\cdot dy=\left(\frac{a}{\pi}\right)\int_{0}^{\infty}dr\int_{0}^{2\pi}d\theta\cdot r\exp\left(-ar^2\right)=$$

$$\left(\frac{a}{\pi}\right)2\pi\int_{0}^{\infty}r\exp\left(-ar^2\right)\cdot dr=\left(\frac{a}{\pi}\right)2\pi\int_{0}^{\infty}\exp\left(-ar^2\right)\cdot\frac{1}{2a}\cdot d\left(ar^2\right)=$$

$$\left(\frac{a}{\pi}\right)\frac{\pi}{a}\int_{0}^{\infty}\exp(-z)\cdot dz=1$$

Normalization confirmed.
(note that “somehow” in the variable r^2 we obtained the exponential distribution instead of the normal distribution. This small calculation can serve as an example of how we can switch from one type of distribution to another. More in the section)

Slide 19: Expectation value of variance – shows that it is always positive
Here is a slight detour. Write

$$E\left(X^2\right)-E^2(X)=E\left(X^2\right)-2E^2(X)+E^2(X)=$$

$$E\left(X^2-2E^2(X)+E^2(X)\right)=E\left(X^2-2X\cdot E(X)+E^2(X)\right)$$

$$E\left[\left(X-E(X)\right)^2\right]\geq 0$$

After Slide 22

SIDE STORY: Volume filled by a diffusing particle.

Consider am unfolded protein molecule is executing a diffusive motion (until it hits a chaperone). A diffusive motion can be modeled as a sequence of uncorrelated displacements, each of them sampled from a normal distribution in three dimensions. Let the displacement of step *i* be $\Delta_i = (\Delta x_i, \Delta y_i, \Delta z_i)$ which is considered as a random vector (three independent random variables). The probability of a particular displacement is given by

$$p(\Delta x_i, \Delta y_i, \Delta z_i) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} \exp\left[-\frac{\Delta_i^2}{2\sigma^2}\right]$$

What is the volume covered by the particle after *N* steps?

It is useful to characterize the volume of a "walk" by the average end to end distance. If X_0 is the position at the beginning (which we arbitrarily set to $(0,0,0)$ and X_N the position by the end of the walk, the end to end distance is

$$|X_N|_2 = \sqrt{\sum_{i=1,\dots,N} \left(\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2\right)} = \sqrt{\sum_{i=1,\dots,N} \Delta_i^2}$$

We will compute the average of the square of the end-to-end distance and considering taking root only after the averaging. This will give an estimate of the size of a distribution of walks. We have

$$\left\langle X_N^2 \right\rangle = \int_{-\infty}^{\infty} \left(\sum_{i=1,\dots,N} \Delta_i^2\right)\left(\frac{1}{2\pi\sigma^2}\right)^{3N/2} \exp\left(-\frac{\sum_{j=1,\dots,N} \Delta_j^2}{2\sigma^2}\right) \prod_{k=1,\dots N} dx_k dy_k dz_k =$$

$$= \sum_{i=1,\dots,N} \int_{-\infty}^{\infty} \left(\Delta_i^2\right)\left(\frac{1}{2\pi\sigma^2}\right)^{3N/2} \exp\left(-\frac{\sum_{j=1,\dots,N} \Delta_j^2}{2\sigma^2}\right) \prod_{k=1,\dots N} dx_k dy_k dz_k$$

$$= \sum_{i=1,\dots N} \int_{-\infty}^{\infty} \left(\Delta_i^2\right)\left(\frac{1}{2\pi\sigma^2}\right)^{3/2} \exp\left(-\frac{\Delta_i^2}{2\sigma^2}\right) dx_i dy_i dz_i =$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} \sum_{i=1,\dots N} \int_0^{\infty} \left(\Delta_i^2\right) \exp\left(-\frac{\Delta_i^2}{2\sigma^2}\right) \cdot 4\pi\Delta_i^2 d\Delta_i$$

As a next step we consider the explicit evaluation of the "core" integral

$$\int_0^{\infty} \Delta^4 \exp\left(-\lambda\Delta^2\right) d\Delta = \frac{\partial^2}{\partial\lambda^2} \int_0^{\infty} \exp\left(-\lambda\Delta^2\right) d\Delta = \frac{\partial^2}{\partial\lambda^2}\left(\frac{\pi}{\lambda}\right)^{1/2} = \frac{3}{4\lambda^2}\left(\frac{\pi}{\lambda}\right)^{1/2}$$

Putting it all together $\left(\lambda = 1/2\sigma^2\right)$

$$\left\langle X_N^2 \right\rangle = \frac{N}{\left(2\pi\sigma^2\right)^{3/2}} 4\pi \frac{3}{4}\left(2\sigma^2\right)^2 \left(2\pi\sigma^2\right)^{1/2} = 3N\left(2\sigma^2\right)$$

The typical end-to-end distance is proportional to $\sqrt{N}$ and not N (the number of steps. It is also does not seem to depend strongly on the dimensionality of the system.

Slide 23: Markov inequality

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) = \int_{-\infty}^{t} x \cdot f(x) dx + \int_{t}^{\infty} x \cdot f(x) dx$$

(since x has only non-negative values)

$$E(X) \geq \int_{t}^{\infty} x \cdot f(x) dx \geq \int_{t}^{\infty} t \cdot f(x) dx = t \cdot \Pr(X \geq t)$$

For $E(X) > t$ (which is not interesting) the right hand side of the equation $\Pr(X \leq t) \leq E(X)/t$ is larger than one. The distribution function is at most one, so no new information here.