Problem 1

A radioactive substance X decays continuously at a constant rate of $3 * 10^{-3}\%$ per year.
What is the probability for a single substace X atom to decay in the next 100 years?

Solution 1

With a constant rate of decay we can express the change in the number of atoms of substance X in a sample of starting size n_0 as $-dN = N * \lambda * dt$, where $\lambda = 3 * 10^{-3}\%$ per year (or just $\lambda = 3 * 10^{-5}\ year^{-1}$).
Like in class, here N is the number of atoms (that didn't decay yet) and dN is the loss of substance X atoms.
From the above equation we can get that $\frac{-dN}{N} = \lambda dt$.
We will integrate both sides. Notice that the left side our bounderies are n_0 and n_T (initial sample size and sample size at time T) while on the right side our boundries are 0 and T.

$$\int_{n_0}^{n_T} \frac{-dN}{N} = \int_0^T \lambda dt \;\Rightarrow\; \int_{n_0}^{n_T} \frac{1}{N} dN = \int_0^T -\lambda dt \;\Rightarrow\; log(N)_{n_0}^{n_T} = -\lambda t]_0^T \Rightarrow$$

$$log(n_T) - log(n_0) = -\lambda(T - 0) \;\Rightarrow\; log(\frac{n_T}{n_0}) = -\lambda T \;\Rightarrow\; \frac{n_T}{n_0} = e^{-\lambda T} \;\Rightarrow\; n_T = n_0 e^{-\lambda T}$$

We are looking for the amount of atoms that decayed which is

$$n_0 - n_T = n_0 - n_0 e^{-\lambda T} = n_0(1 - e^{-\lambda T})$$

The fraction of the sample decayed in time T can be expressed as

$$\frac{n_0 - n_T}{n_0} = \frac{n_0(1 - e^{-\lambda T})}{n_0} = (1 - e^{-\lambda T})$$

Using the values from the question we get that in $T = 100\ years$ and a constant decay rate of $\lambda = 3 * 10^{-5}\ year^{-1}$ the fraction that decays is $(1 - e^{-3*10^{-5}*100})$.
So the probability of a single atom from the sample to decay in this time is $(1 - e^{-3*10^{-3}}) \approx -3 * 10^{-3}$.

Problem 2

Using a Uniform random variable U on the interval $(0, 10)$, how would you express
The Uniform random variable Y on the interval (-1,1)?
The Exponential random variable W with parameter $\lambda = 2$?

Solution 2

If U is Uniform on $(0, 10)$ then $\frac{U}{5}$ is Uniform on $(0, 2)$ and $Y = \frac{U}{5} - 1$ is Uniform on $(-1, 1)$.
We saw in section that if F is a c.d.f and there exists an inverse function F^{-1} then for a Uniform $(0, 1)$ random variable V, the new formed random variable $X = F^{-1}(V)$ has the p.d.f. F.
In this problem we are interested in a r.v. W with the c.d.f. $F_W(w) = 1 - e^{\lambda w} = 1 - e^{2w}$.
The function F^{-1} would then be $\frac{-log(1-V)}{\lambda} = \frac{-log(1-V)}{2}$.
But U is Uniform $(0, 10)$, so $V = \frac{U}{10}$ should be Uniform $(0, 1)$ and our desired random variable will be

$$W = \frac{-log(1 - \frac{U}{10})}{2}$$