Solutions

1. True/False [20 pts] (parts a–j)

   Each correct answer is 2 pts; each wrong answer is -2 pts; and each blank answer is 0 pts.

   (a) Software testing proves the presence of bugs, but cannot prove their absence.

   True

   (b) The type that SML infers for the expression `fn (x,y) => fn x => (y,x)` is: `'a * 'b -> 'c -> 'b * 'c`.

   True

   (c) The function $f(n) = \lg(n \ \lg n)$ is $O(\lg n)$.

   True

   (d) At each collection, a copying collector must traverse all of the data in the program (including data unreachable from the roots).

   False

   (e) The implementation of Dijkstra's shortest-paths algorithm requires a stack data structure.

   False

   (f) The function `foldl` is tail-recursive.

   True

   (g) It is possible that a hash table with n elements and a load factor of 2 has a bucket that contains all of the n elements.

   True

   (h) When a program exhibits temporal locality, it will access the same memory location in the near future.

   True

   (i) Any lookup operation in a splay tree with n nodes is $O(\lg n)$.

   False

   (j) Data races can occur during the execution of message-passing concurrent programs.

   False

2. Sets [20 pts] (parts a–c)

The following is a standard set interface:

```
signature SET = sig
 (* A 'a set is a set of items of type 'a. *)
 type 'a set

 (* empty is the empty set *)
 val empty : 'a set
 (* add(s,e) is s union {e} *)
 val add: 'a set * 'a -> 'a set
 (* fold over the elements of the set *)
 val fold: ('a*'b->'b) -> 'b -> 'a set -> 'b
end
```

(a) [5 pts] Extend the interface with a function `remove` that removes an item from a set. Provide a signature *and* a specification for `remove`. Define an appropriate exception if necessary.

Answer:

```
exception NotFound

(* remove (s,e) is the set s - {e} *)
 * Raises: NotFound if e does not belong to s *)
 val remove: 'a set * 'a -> 'a set
```

(b) [7 pts] Write an implementation for function `remove` using the other functions in the signature. Assume that an equality funtion for items `equal':'a*'a->bool` is also available (`'a` being the type of the items in the set). Your function `remove` should not visit any of the items more than once.

Answer:

```
fun remove(s, e) =
  let val (r,f) = fold (fn (x,(s',f)) => if equal(x,e) then (s',true)
                                         else (add(s',x),f))
                       (empty,false) s
  in
    if f then r else raise NotFound
  end
```

(c) [8 pts] Consider now a function `cartprod` that takes two sets of items and yields a set of pairs representing the Cartesian product:

```
(* cartprod(s1,s2) is the cartesian product of s1 and s2 *)
val cartprod: 'a set * 'a set -> ('a * 'a) set
```

Remember that the Cartesian product $A \times B$ of two sets A and B is the set of all pairs (a, b) where $a \in A$ and $b \in B$. That is, $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$.

For simplicity, assume that sets are implemented using lists (`type 'a set = 'a list`). Below are some examples of using `cartprod`:

```
cartprod ([1,2], [3,4]) = [(1,3), (1,4), (2,3), (2,4)]
cartprod (["a"], ["b", "c"]) = [("a","b"), ("a","c")]
cartprod ([1,2], []) = []
```

Write the function `cartprod`, assuming a list implementation of sets. You *may not* use the list concatenation operator "`@`" in your solution.

Note: It is possible to write `cartprod` such that is works for any implementation of sets, not only lists. Feel free to write such a function.

Answer: The implementation of `cartprod` for lists, without folding:

```
fun cartprod(s1, s2) =
  case (s1,s2) of
    ([],_) => []
  | (h1::t1, _) =>
      let fun prod(s) = case s of
            [] => cartprod(t1,s2)
          | h2::t2 => (h1,h2)::prod(t2)
      in
        prod(s2)
      end
```

The general implementation of `cartprod` using `fold` is simpler:

```
fun cartprod(s1, s2) =
  fold (fn (x,s') =>
          (fold (fn (y,s'') => add (s'',(x,y))) s' s2))
       empty s1
```

3. Trees [20 pts] (parts a–c)

The following is the standard datatype for binary search trees containing integer values:

```
datatype tree = Leaf | Node of tree * int * tree
```

(a) [5 pts] Consider two functions `min` and `max` that compute the smallest and the largest numbers in a tree:

```
(* min(t) is the smallest element of t,
 *        or the largest integer if t is a leaf. *)
val min : tree -> int
(* max(t) is the largest element of t,
 *        or the smallest integer if t is a leaf. *)
val max : tree -> int
```

Using these functions, write a function `repOK : tree -> bool` that returns true if and only if the tree satisfies the binary search tree invariant. (For an informal description of the invariant you'll receive partial credit).

Answer:

```
fun repOK(t) =
  case t of Leaf => true
          | Node(l, v, r) => max(l) <= v andalso v <= min(r)
                             andalso repOK(l) andalso repOK(r)
```

(b) [5 pts] Several kinds of binary trees (including AVL, red-black, and splay trees) use rotations for rebalancing. The following is the basic right rotation:

```
fun rotate (t:tree) :tree =
    case t of
      Node(Node(A,x,B), y, C) => Node(A, x, Node(B,y,C))
    | _ => t
```

Show that the above function `rotate` maintains the binary search tree invariant.

Answer: *Assume that the invariant holds for* `t` *at the beginning of* `rotate`.

On the first arm pf the case construct, `t` *matches the pattern* `Node(Node(A,x,B),y,C)`. *Therefore,* $\texttt{max(A)} \le \texttt{x} \le \texttt{min(B)} \le \texttt{max(B)} \le \texttt{y} \le \texttt{min(C)}$. *Also, all of the nodes in* `A`, `B`, *and* `C` *satisfy the binary search tree invariant. This shows that the tree* `Node(A,x,Node(B,y,C))` *is also a binary search tree.*

On the second arm of the case, the invariant trivially holds because the returned tree is identical.

(c) [10 pts] Consider now an imperative implementation of binary search trees:

```
datatype itree = Leaf | Node of (itree ref) * int * (itree ref)

(* irotate(t) performs a right rotation at the root of t
 * Effects: destructively updates t
 * Returns: the new root after the rotation. *)
val irotate: itree -> itree
```

Write an implementation for `irotate`.

Answer:

```
val irotate(ty:itree): itree =
  case ty of
    Node(L as ref(tx as Node(A,x,B)), y, C) =>
        (L := !B; B := ty; tx)
  | _ => ty
```

4. Correctness and Complexity [20 pts] (parts a–g)

Consider the following program:

```
fun f (x, y) =
  if y = 0 then 1 else
    let
      val p = f (x, y div 2)
      val sp = p * p
    in
      if y mod 2 = 0 then sp else x * sp
    end
```

(a) [3 pts] What does `f(x,y)` compute?

Answer: `f(x,y)` *is* $\mathtt{x}^{\mathtt{y}}$.

The following questions ask you to prove the correctness of function `f`, with respect to your answer above.

(b) [2 pts] Write the property $P(n)$ that you need to prove and specify the initial value n_0 of n.

Answer: $P(n) = f(x,n)$ *is* x^n*, for all* x*. The initial value of* n *is* $n_0 = 0$.

(c) [2 pts] State whether you'll use strong or weak induction.

Answer: *Strong induction.*

(d) [2 pts] Prove the base case.

Answer: *For* $n = 0$ *and for any* x, $f(x,n) = f(x,0)$ *evaluates to 1. Since* $x^0 = 1$*, we get that* $f(x,0) = x^0$ *for all* x.

(e) [4 pts] State the induction hypothesis and prove the induction step.

Answer: *Induction Hypothesis: assume that* $P(m)$ *holds for any* $0 \leq m < n$*. We want to prove that* $P(n)$ *holds.*

Since $n > 0$, $P(n)$ *evaluates the false arm of the first if statement. On that branch, the program computes* $p = f(x, \lfloor n/2 \rfloor)$ *and* $sp = p^2$*. By IH, because* $\lfloor n/2 \rfloor < n$*, we get that* $p = x^{\lfloor n/2 \rfloor}$*. Hence,* $sp = x^{2\lfloor n/2 \rfloor}$*. Note that* $2\lfloor n/2 \rfloor$ *is not necessarily equal to* n*, because* $\lfloor n/2 \rfloor$ *is the integer division.*

We have two cases. If $n \bmod 2 = 0$*, the program executes the first branch of the inner if expression. In this case,* $n = 2k$ *for some* k*, so* $p = x^k$ *and* $sp = p^2 = x^{2k} = x^n$*. The returned value is sp, so* $f(x,n) = x^n$.

If $n \bmod 2 = 1$*, the program executes the second branch. In this case,* $n = 2k+1$ *for some* k*, so* $p = x^k$ *and* $sp = p^2 = x^{2k} = x^{n-1}$*. The returned value is* $x * sp$*, so* $f(x,n) = x * sp = x^n$.

In either case, $f(x,n) = x^n$*, which completes the proof.*

Next, analyze the run-time complexity of `f`.

(f) [4 pts] Write the recurrence relations for the running time of `f(2,n)`. Use constants c_1, c_2, etc. for operations that take constant time.

Answer: *Let* $T(n)$ *be the running time of* $f(2,n)$*. Then:*

$$\begin{aligned} T(0) &= c_1 \\ T(n) &= T(\lfloor n/2 \rfloor) + \begin{cases} c_2 & \text{if } n \text{ is even} \\ c_2 + c_3 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

(g) [3 pts] What is the run-time complexity of `f(2,n)`? You don't have to prove your result.

Answer: $T(n)$ *is* $O(\lg n)$.

5. Environment Model [20 pts] (parts a–d)

The program below is written in a ML-like language that doesn't allow recursive functions, but has references and higher-order functions:

```
let val rf = ref (fn x => x)
    val f = fn y => let val (n,p,q) = y in
```

```
                    if n = 0 then y else
                    (!rf)(n - 1, q, p + q)
                    end
    val () = rf := f
in
    f(5,1,1)
    (* GC *)
end
```

(a) [3 pts] What are the types of `f` and `rf`?

Answer:

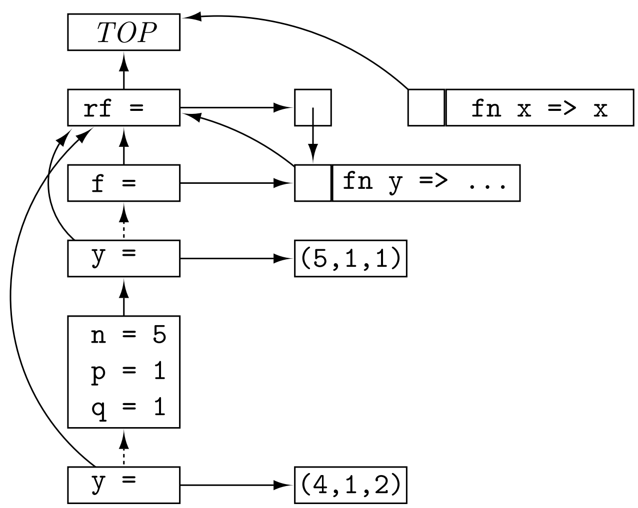
```
f :  int * int * int -> int * int * int
rf :  (int * int * int -> int * int * int) ref
```

(b) [3 pts] What is the result of the evaluation?

Answer: `(0,8,13)`.

(c) [10 pts] Draw the environment diagram that arises during the evaluation of the second call to `f` (i.e., when the program starts evaluating the function body for that call).

Answer:



(d) [4 pts] What heap cells (not environment entries!) can a garbage collector reclaim at the program point labeled `GC` in the code?

Answer: *The collector can reclaim the closure* `fn x => x`*, as well as five tuples created during the execution:* `(5,1,1)`*,* `(4,1,2)`*,* `(3,2,3)`*,* `(2,3,5)`*, and* `(1,5,8)`*.*

The cells that cannot be collected are: the closure `fn y => ...`*, the ref cell, and the returned tuple* `(1,8,13)`*. The first two cannot be reclaimed yet because they are still reachable from variables in scope.*