CS 3110

Balanced Trees

Nate Foster
Spring 2019

Today's music: Get the Balance Right by Depeche Mode

Review

Previously in 3110:

- Streams

Today:

- Balanced trees

Running example: Sets

```
module type Set = sig
  type 'a t
  val empty  : 'a t
  val insert : 'a -> 'a t -> 'a t
  val mem    : 'a -> 'a t -> bool
  ...
end
```

Set implementations: performance

	Workload 1	
	`insert`	`mem`
`ListSet`	35s	106s

MacBook, 1.3 GHz Intel Core m7, 8 GB RAM, median of three runs

Set implementations: performance

	Workload 1	
	`insert`	`mem`
`ListSet`	35s	106s
`BstSet`	130s	149s

MacBook, 1.3 GHz Intel Core m7, 8 GB RAM, median of three runs

Set implementations: performance

	Workload 1		Workload 2	
	`insert`	`mem`	`insert`	`mem`
`ListSet`	35s	106s	35s	106s
`BstSet`	130s	149s	0.07s	0.07s

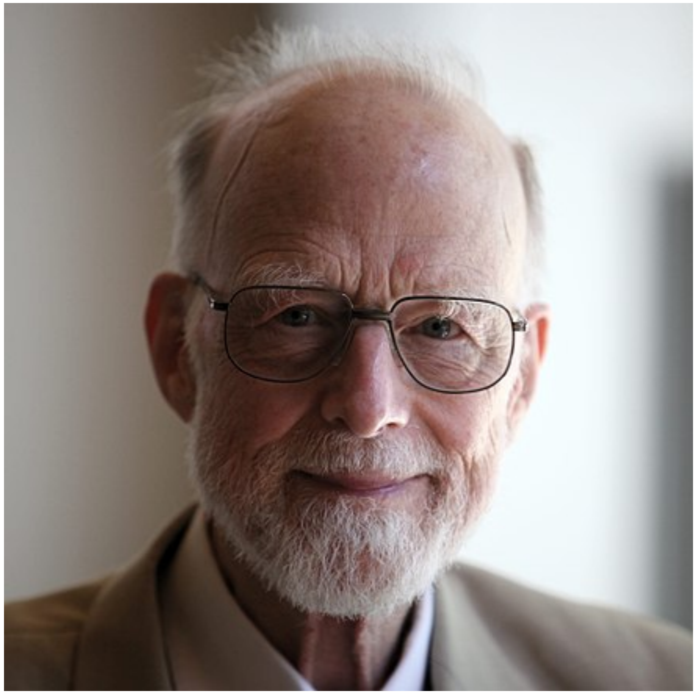
MacBook, 1.3 GHz Intel Core m7, 8 GB RAM, median of three runs

Set implementations: performance

	Workload 1		Workload 2	
	`insert`	`mem`	`insert`	`mem`
`ListSet`	35s	106s	35s	106s
`BstSet`	130s	149s	0.07s	0.07s
`RbSet`	0.12s	0.07s	0.15s	0.08s

MacBook, 1.3 GHz Intel Core m7, 8 GB RAM, median of three runs

Sir Tony Hoare



b. 1934

Turing Award Winner 1980

For his fundamental contributions to the definition and design of programming languages.

"We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil."

LIST SET

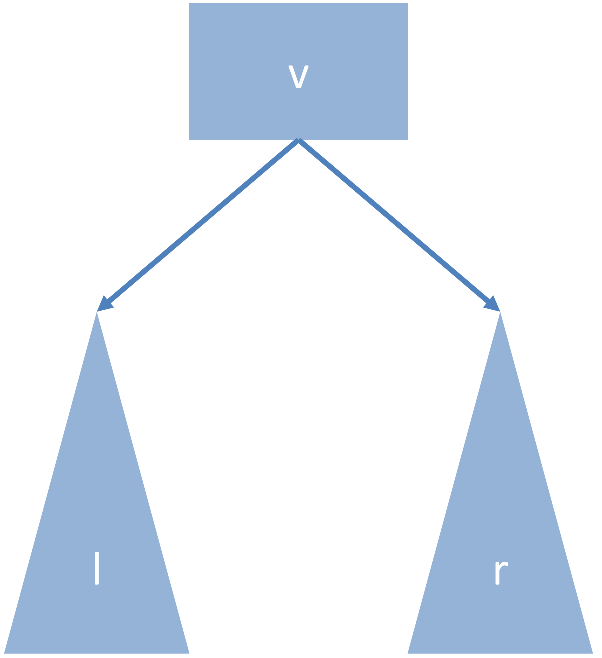
Dem
o

BST SET

Dem
o

Binary search tree (BST)

- Binary tree: every node has two subtrees
- BST invariant:
  - all values in l are less than v
  - all values in r are greater than v



Back to performance

	Workload 1		Workload 2	
	`insert`	`mem`	`insert`	`mem`
`ListSet`	35s	106s	35s	106s
`BstSet`	130s	149s	0.07s	0.07s

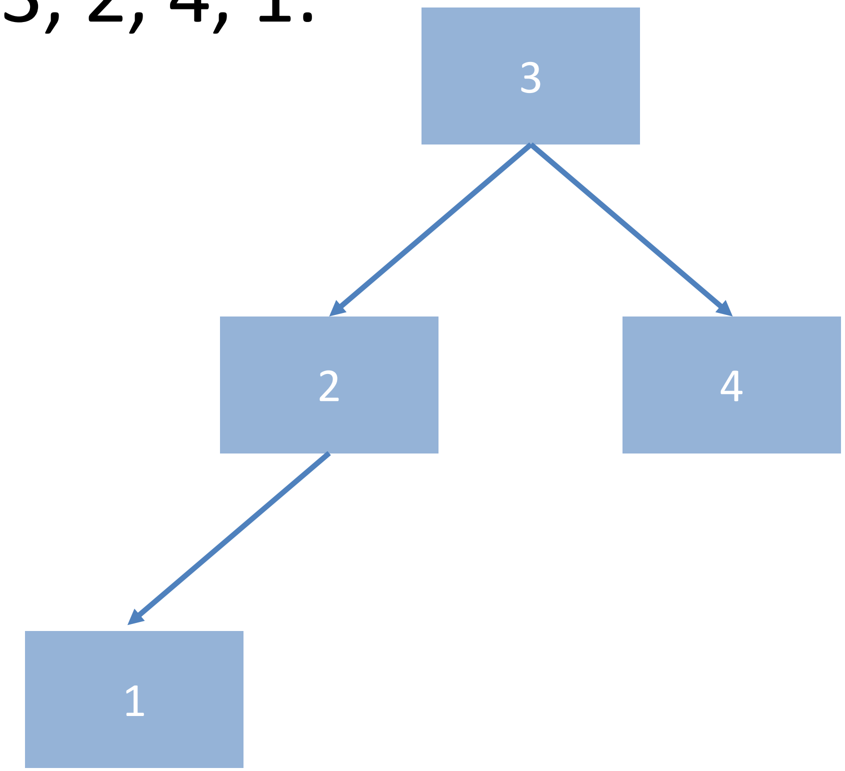
Workloads

- Workload 1:
  - `insert`: 50,000 elements in ascending order
  - `mem`: 100,000 elements, half of which not in set

- Workload 2:
  - `insert`: 50,000 elements in random order
  - `mem`: 100,000 elements, half of which not in set

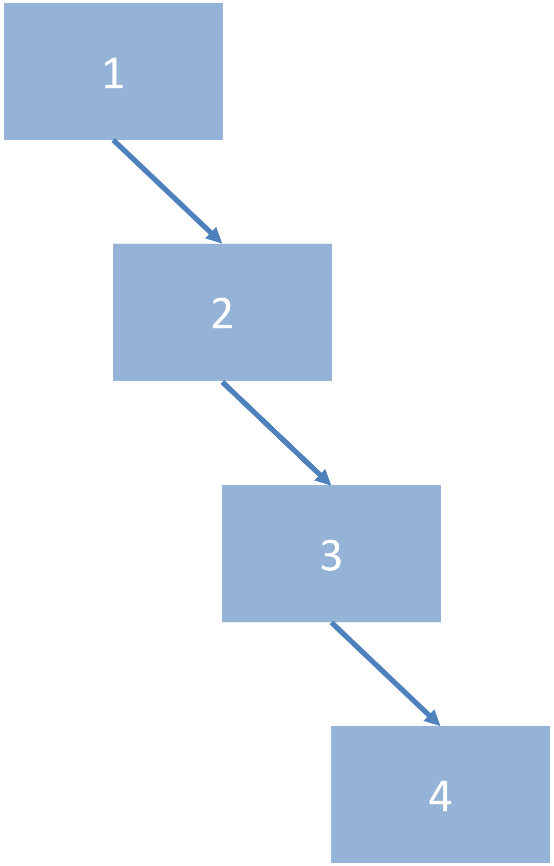
Insert in random order

- Resulting tree depends on exact order
- One possibility for inserting 1..4 in random order 3, 2, 4, 1:



Insert in linear order

Only one possibility for inserting 1..4 in linear order 1, 2, 3, 4:

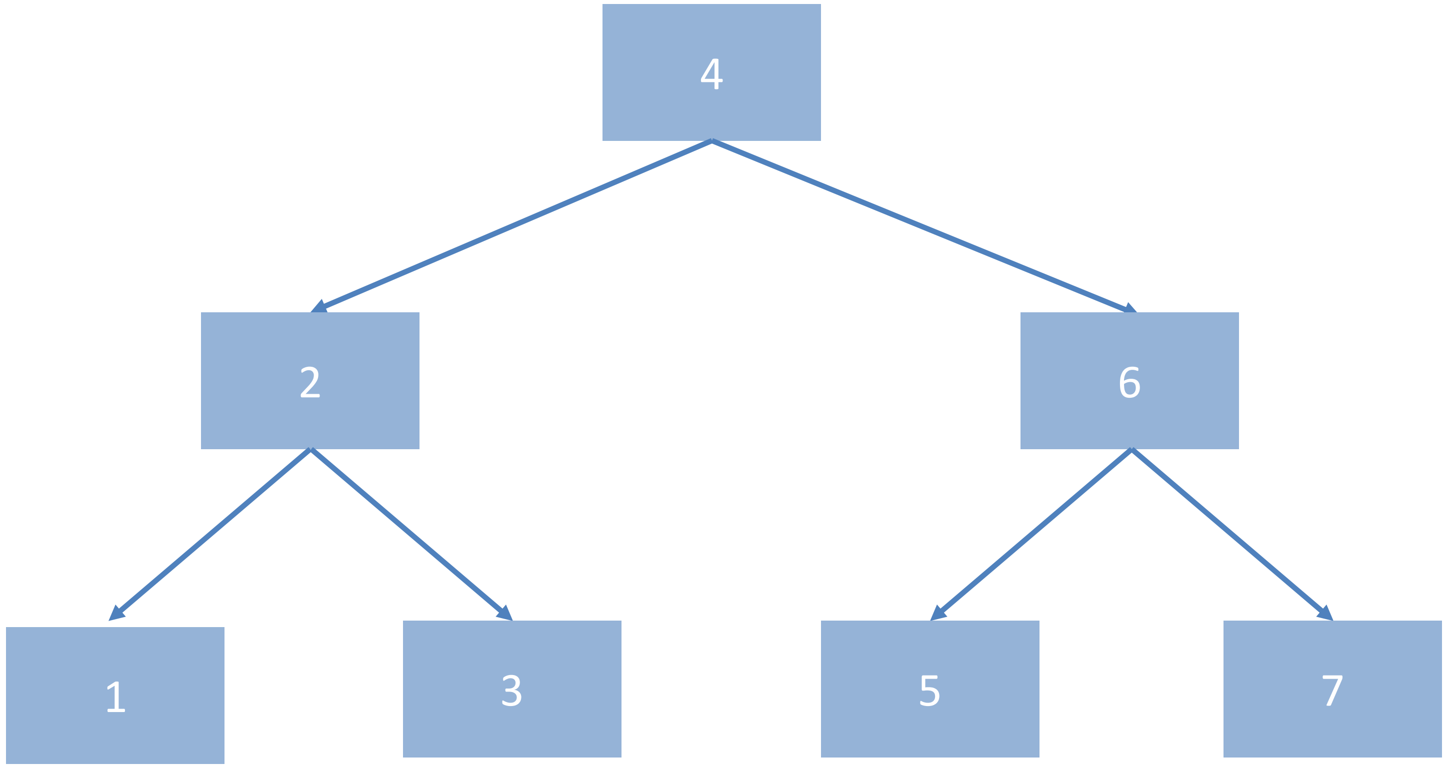


unbalanced: leaning toward the right

When trees get big

- Inserting next element in linear tree always takes *n* operations where *n* is number of elements in tree already
- Inserting next element in randomly-built tree might take far fewer…

Best case tree



all paths through *perfect binary tree* have same length: $log_2\ (n+1)$,
where *n* is the number of nodes,
recalling there are implicitly leafs below each node at bottom level

Performance of BST

- `insert` and `mem` are both O(n)
- But if trees always had short paths instead of long paths, could be better: O(log n)
- How could we ensure short paths?
  i.e., *balance* trees so they don't lean

Strategies for achieving balance

- In general:
  - Strengthen the RI to require balance
  - And modify insert to guarantee it
- Well known data structures:
  - 2-3 trees: all paths have same length
  - AVL trees: length of shortest and longest path from any node differ at most by one
  - Red-black trees: length of shortest and longest path from any node differ at most by factor of two
- All of these achieve O(log(n)) `insert` and `mem`

RED-BLACK TREES

Red-black trees

- [Guibas and Sedgewick 1978], [Okasaki 1998]
- Binary search tree with:
  - Each node colored red or black
  - Leafs colored black
- RI: BST +
  - Local invariant: No red node has a red child
  - Global invariant: Every path from the root to a leaf has the same number of black nodes

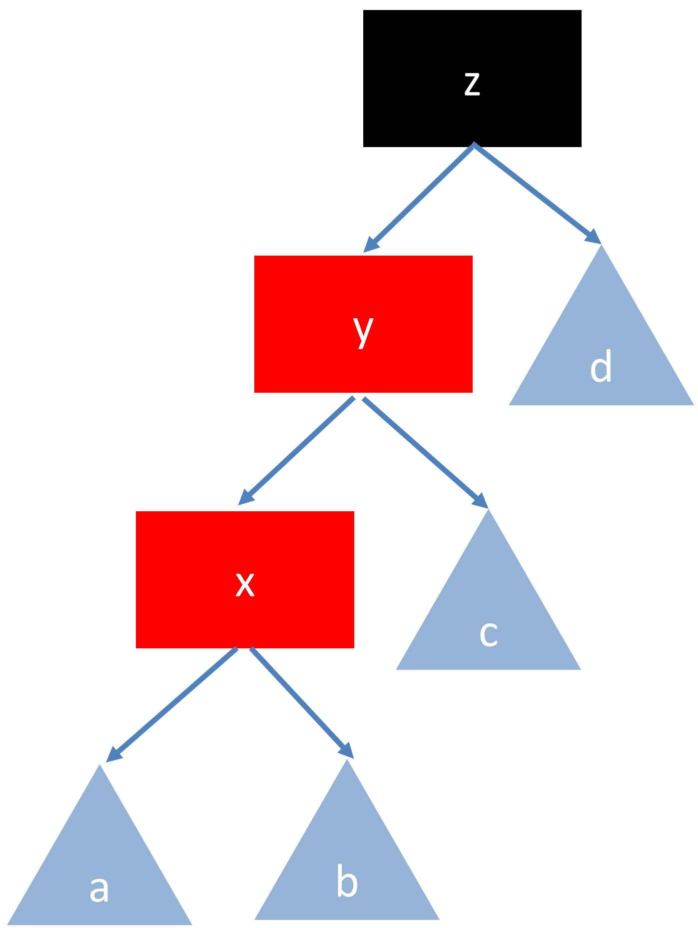
Path length

- Invariants:
  - No red node has a red child
  - Every path from the root to a leaf has the same number of black nodes
- Together imply: length of longest path is at most twice length of shortest path
  - e.g., B-R-B-R-B-R-B vs. B-B-B-B

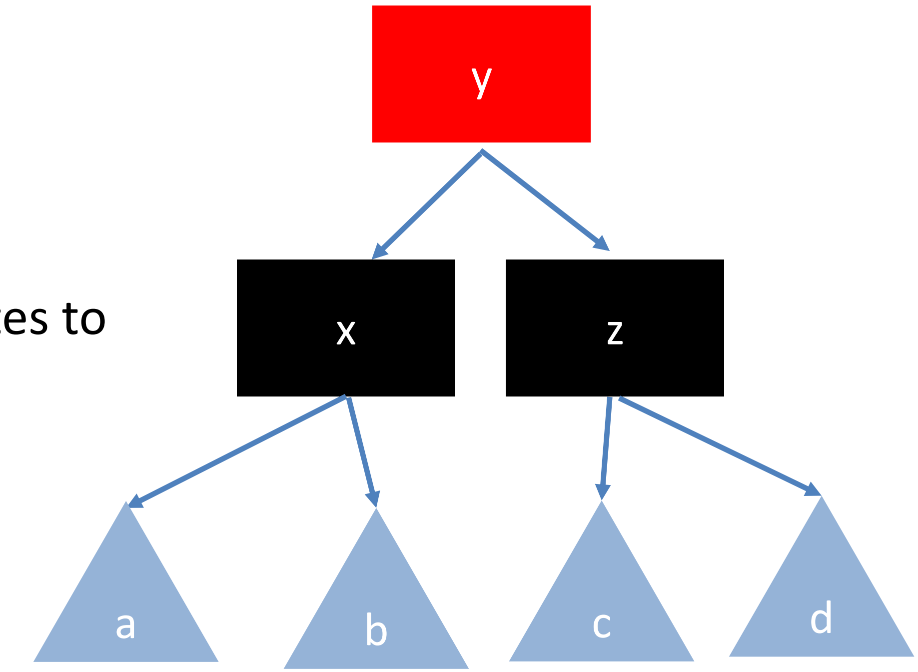
RED-BLACK SET

Dem
o

RB rotate (1 of 4)

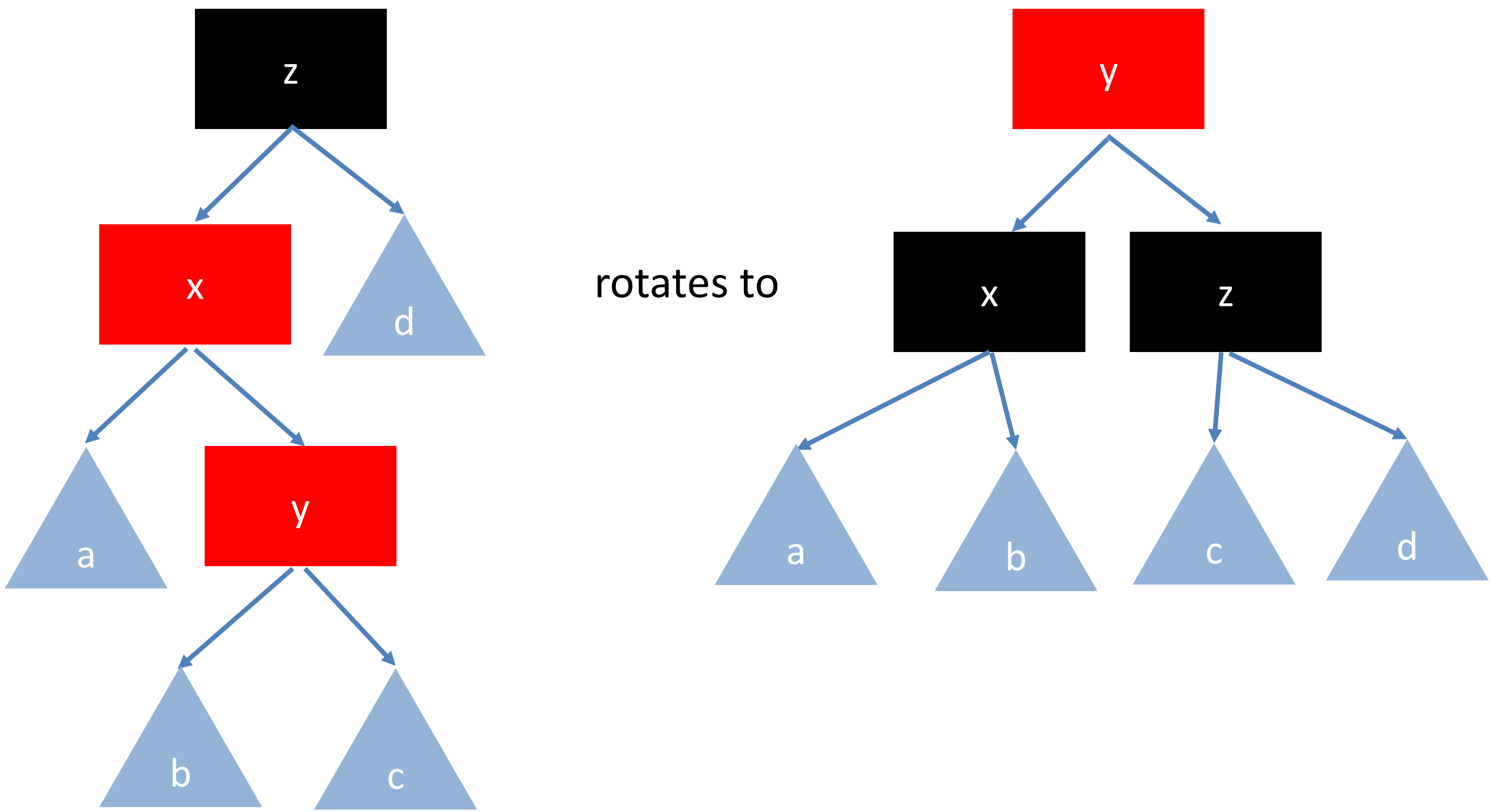


rotates to

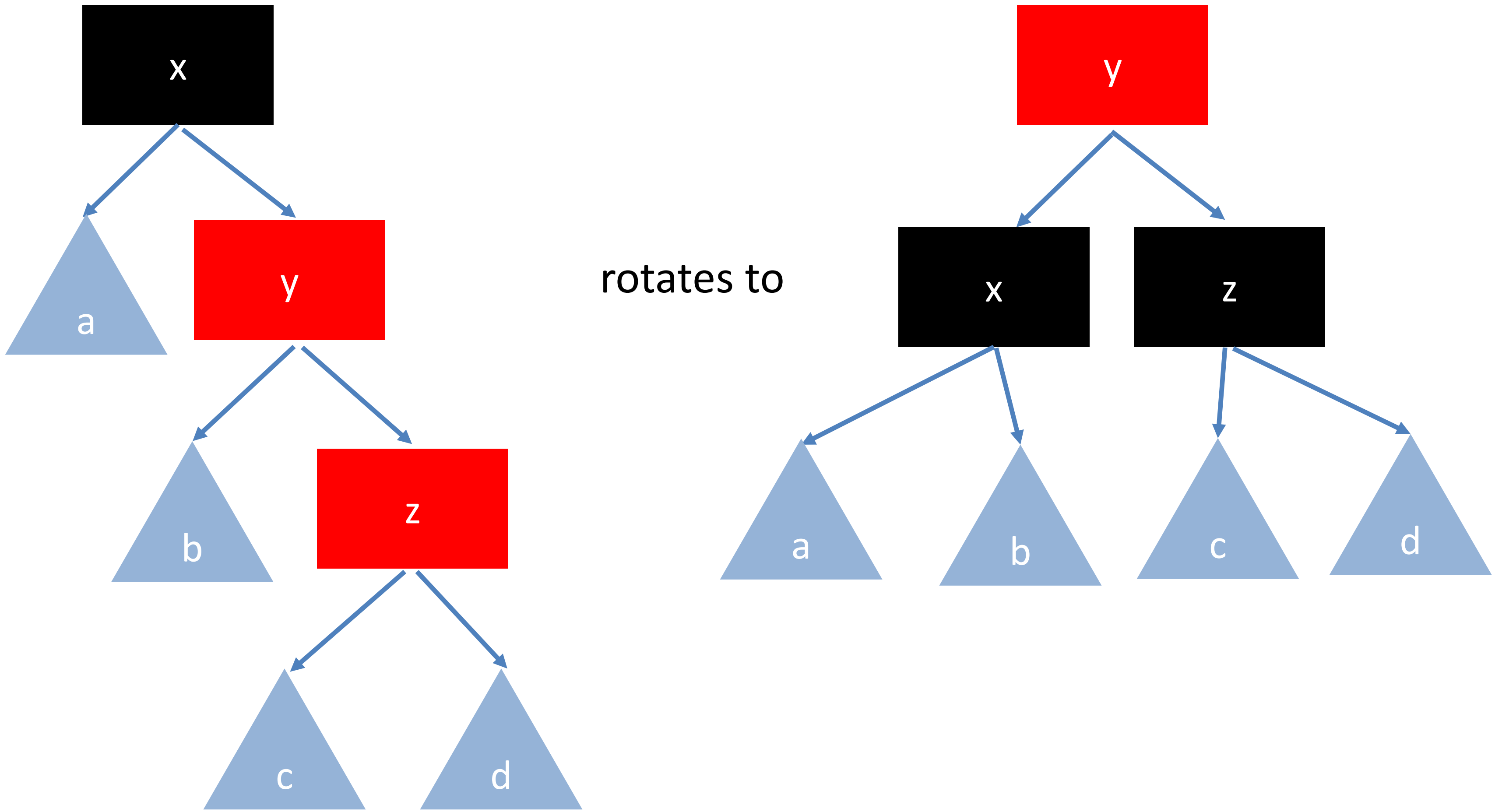


eliminates y-x violation
but maybe y has a red parent: new violation
keep recursing up tree

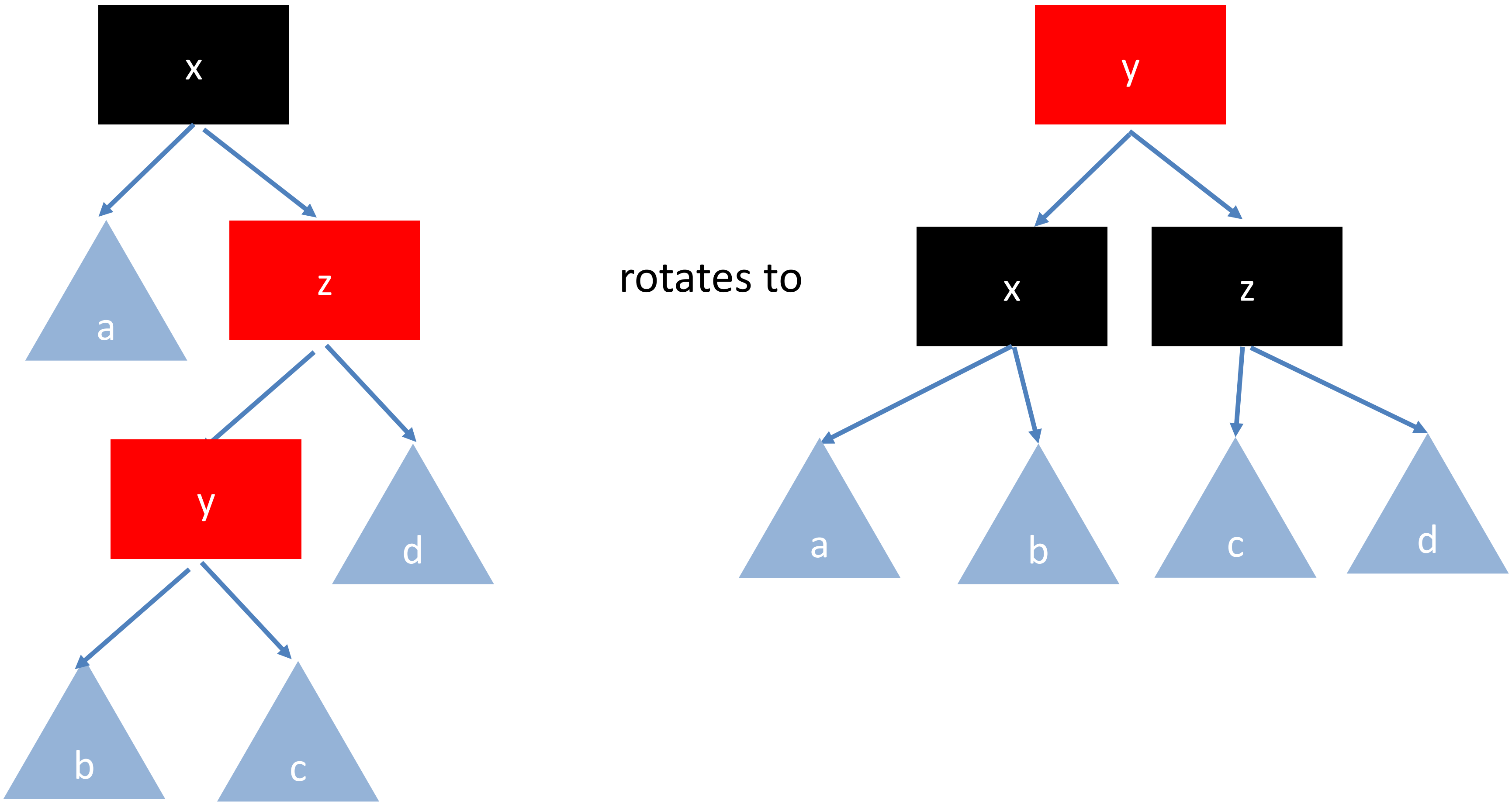
RB rotate (2 of 4)



RB rotate (3 of 4)



RB rotate (4 of 4)



RB balance

```
let balance = function
  | (Blk, Node (Red, Node (Red, a, x, b), y, c), z, d) (* 1 *)
  | (Blk, Node (Red, a, x, Node (Red, b, y, c)), z, d) (* 2 *)
  | (Blk, a, x, Node (Red, Node (Red, b, y, c), z, d)) (* 4 *)
  | (Blk, a, x, Node (Red, b, y, Node (Red, c, z, d))) (* 3 *)
    -> Node (Red, Node (Blk, a, x, b), y, Node (Blk, c, z, d))
  | t -> Node t
```

Upcoming events

- [Today] Foster Office Hours 1:15-2:15
- [Tonight] Level up!
- [Tuesday] Prelim
- [next Thursday] Guest lecture: Dean Morrisett

This is blissfully balanced.

WE ARE GROOT