CS 3110

Induction in Coq

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Fall 2018

Today's music: *Pictures of Pandas Painting* by They Might Be Giants

Attendance question

Using the "propositions as types" correspondence, what proposition does this program prove?

```
let rec loop x = loop x
```

A. P
B. $P \Rightarrow P$
C. $P = P$
D. $P \Rightarrow Q$
E. $(P \Rightarrow Q) \Rightarrow R$

Review

Previously in 3110:

- Functional programming in Coq
- Logic in Coq
- Proofs are programs (Curry-Howard, BHK)

Today:

- Induction in Coq

INDUCTION ON NATURAL NUMBERS

Structure of inductive proof

Theorem:
`for all natural numbers n, P(n).`

Proof: `by induction on n`

Case: `n = 0`
Show: `P(0)`

Case: `n = k+1`
IH: `P(k)`
Show: `P(k+1)`

QED

Sum to n

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

$$\sum_{i=0}^{n} i$$

Theorem:

```
for all natural numbers n,
  sum_to n = n * (n+1) / 2.
```

Proof: `by induction on n`

Discussion: What is P? Base case? Inductive case? Inductive hypothesis?

Proof

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

`P(n) ≡ (sum_to n = n * (n+1) / 2)`

Case: `n = 0`

Show:

`P(0)`

Case: `n = k+1`

IH: `P(k) ≡ sum_to k = k * (k+1) / 2`

Show:

`P(k+1)`

QED

INDUCTION ON LISTS

Structure of inductive proof

Theorem:
`for all natural numbers n, P(n).`

Proof: `by induction on n`

Case: `n = 0`
Show: `P(0)`

Case: `n = k+1`
IH: `P(k)`
Show: `P(k+1)`

QED

Structure of inductive proof

Theorem:
`for all lists lst, P(lst).`

Proof: `by induction on lst`

Case: `lst = []`
Show: `P([])`

Case: `lst = h::t`
IH: `P(t)`
Show: `P(h::t)`

QED

Append nil

```
let rec (@) lst1 lst2 =
  match lst1 with
  | []   -> lst2
  | h::t -> h :: (t @ lst2)
```

Theorem:

`for all lists lst, lst @ [] = lst.`

Proof: `by induction on lst`

Discussion: What is P? Base case? Inductive case? Inductive hypothesis?

```
let rec (@) lst1 lst2 =
  match lst1 with
  | []   -> lst2
  | h::t -> h :: (t @ lst2)
```

Base case

`P(lst) ≡ lst @ [] = lst`

Case: `lst = []`

Show:

`P([])`

Case: `lst = h::t`

IH: `P(t) ≡ t @ [] = t`

Show:

`P(h::t)`

QED

INDUCTION ON LISTS IN COQ

Demo

INDUCTION ON NATS

Inductive types

`induction` works on inductive types, e.g.

```
Inductive list (A : Type) : Type :=
  | nil : list A
  | cons : A -> list A -> list A
```

Need an inductive definition of natural numbers…

Naturals

```
Inductive nat : Set :=
   | O : nat                (* zero *)
   | S : nat -> nat         (* succ *)

type nat = O | S of nat

0 is O
1 is S O
2 is S (S O)
3 is S (S (S O))
```

- unary representation
- Peano arithmetic

Induction on nat(ural)s

Theorem:

`for all n:nat, P(n)`

Proof: `by induction on n`

Case: `n = O`

Show: `P(O)`

Case: `n = S k`

IH: `P(k)`

Show: `P(S k)`

QED

Theorem:

`for all naturals n, P(n)`

Proof: `by induction on n`

Case: `n = 0`

Show: `P(0)`

Case: `n = k+1`

IH: `P(k)`

Show: `P(k+1)`

QED

Goal: redo this proof in Coq

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

$$\sum_{i=0}^{n} i$$

Theorem:

```
for all natural numbers n,
  sum_to n = n * (n+1) / 2.
```

Proof: `by induction on n`

Demo

CONTROLLED RECURSION

Demo

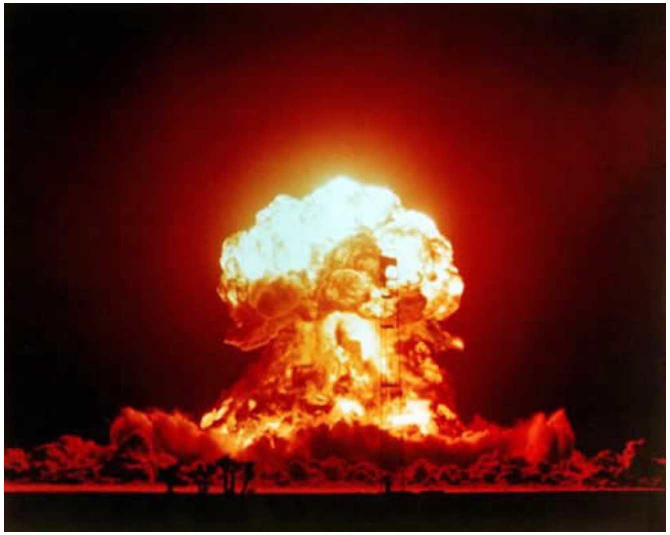
Why no infinite loops?

In OCaml:

```
# let rec loop x = loop x
val loop : 'a -> 'b = <fun>
```

By propositions-as-types, these are the same:

- `'a -> 'b`
- $A \Rightarrow B$



What if A=True, B=False?

Infinite loops prove False!

CONTROLLED RECURSION

Demo

Induction and recursion

- Intense similarity between inductive proofs and recursive functions on variants
  - In proofs: one case per constructor
  - In functions: one pattern-matching branch per constructor
  - In proofs: uses IH on "smaller" value
  - In functions: uses recursive call on "smaller" value

- Proofs = programs
- Inductive proofs = recursive programs

Upcoming events

- **A10 GIST:** tonight, 8 pm, Gates 122

This is inductive.

THIS IS 3110