On the domains of definition of functions

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(*1927*)

In a series of papers published from 1918 onward, Brouwer set forth an intuitionistic "set theory" and on this basis an intuitionistic reconstruction of point-set topology and analysis. The text below is part of a paper published in 1927 (received for publication on 28 April 1926) and contains the proof that every function that is (in the intuitionistic sense) everywhere defined on the closed interval [0, 1] of the continuum is uniformly continuous (Theorem 3). In the course of the argument Brouwer proves the fundamental theorem on "sets" that he later (*1953*) called the bar theorem, as well as its corollary, the fan theorem (Theorem 2).[a]

The text brings together and reworks previous expositions of these results. The uniform-continuity theorem had been asserted earlier (*1923*, p. 4), with only an indication of the fan theorem. The bar theorem and the fan theorem were proved, again for the sake of uniform continuity, in a subsequent paper (*1924* (or its German translation, *1924a*), amended and added to in *1924b* (or *1924c*)).

The intuitionistic theory of the continuum is based on Brouwer's own notion of set (see below, p. 453). Brouwer was dissatisfied, it seems, with the treatment of the continuum by earlier constructivist mathematicians. They either abandoned their constructivism at this point and adopted an axiom of completeness or, as is done in theories of the continuum based on ramified type theory, rejected any means of quantifying over more than a denumerable subset of real numbers at a time (see *Brouwer 1952*, p. 140, and *1953*, p. 1). So what was required was an intuitionistic interpretation of quantification over all sequences of natural numbers or over all sequences satisfying some condition.

To explain the classical conception of such quantification, one sometimes pictures an arbitrary sequence as that which results from one choice for each term, these infinitely many choices being conceived *sub specie aeternitatis*, so that questions about the sequence as a whole (such as whether for some n the nth term is zero) are always objectively determined. Brouwer's idea was to substitute for this the picture of an *infinitely proceeding sequence* of choices that is such that at the nth choice one could restrict one's freedom as to future choices by laying down some (not necessarily deterministic) law. This is presented as a process in time: only so much about the sequence is determined at a given stage of its generation as follows from what the initial segment up to that stage is and from the laws that have been laid down. If there are *no* such laws, nothing will be true of the sequence but what is determined to be true on the basis of some

[a] The earliest printed use of "fan theorem" is, it seems, in *Brouwer 1952*, p. 143; Brouwer used "waaierstelling" earlier in lectures.

initial segment of it. This means that functions whose arguments are free-choice sequences will be continuous.

The conception that gives mathematical form to this picture is Brouwer's notion of *set*, or *spread*.[b] The definition can be expressed as follows (see *Heyting 1956*, pp. 34–35). We have a law Λ_M that characterizes certain finite sequences of natural numbers as *admissible* for the spread M and is such that

(1) Every finite sequence of natural numbers is either admissible or not (this means that the law enables us to *decide* of a given sequence whether it is admissible or not);[c]

(2) If $\langle a_1, \ldots, a_{n+1}\rangle$ is admissible, so is $\langle a_1, \ldots, a_n\rangle$;

(3) An admissible sequence of length 1 can be specified;

(4) If $\langle a_1, \ldots, a_n\rangle$ is admissible, either an m can be found so that $\langle a_1, \ldots, a_n, m\rangle$ is admissible or there is no such m (termination of the process).

Then we have a second law Σ_M that to each sequence admissible for M assigns a definite mathematical object.[d]

Thus, given a sequence $a_1, a_2, \ldots$ of natural numbers such that, for every n, $\langle a_1, \ldots, a_n\rangle$ is admissible, we obtain a corresponding sequence $\xi_1 = \Sigma(\langle a_1\rangle)$, $\xi_2 = \Sigma(\langle a_1, a_2\rangle), \ldots$. A sequence such as $\xi_1, \xi_2, \ldots$ is what Brouwer calls an *element* of the spread M.[e]

As an example, we consider the spread of *points of the continuum*, discussed in the paper below. There are many ways of defining a spread that corresponds to the intuitive notion of the continuum. Brouwer defines a point of the continuum as an infinitely proceeding sequence of intervals that are of the form $I_{m,n} = [m/2^n, (m+2)/2^n]$ and are such that each one lies in the interior of its predecessor. The spread of such points could be defined formally as follows. Let $p_1, p_2, \ldots$ be an enumeration of the pairs of natural numbers, and let $\rho_1(x)$ and $\rho_2(x)$ be such that, if $p_i = \langle r, s\rangle$, $r = \rho_1(i)$ and $s = \rho_2(i)$. Then any sequence $\langle n\rangle$ of length 1 is admissible and is assigned the interval $I_{\rho_1(n),\rho_2(n)}$. If $\langle a_1, \ldots, a_n\rangle$ is admissible and is assigned the interval $I_{r,s}$, then $\langle a_1, \ldots, a_n, k\rangle$ is admissible if and only if

$$\frac{r}{2^s} < \frac{\rho_1(k)}{2^{\rho_2(k)}} < \frac{\rho_1(k)+2}{2^{\rho_2(k)}} < \frac{r+2}{2^s},$$

and then the interval $I_{\rho_1(k),\rho_2(k)}$ is assigned to $\langle a_1, \ldots, a_n, k\rangle$.

Brouwer says that a spread is *finitary* if for every n there can be determined a k_n such that the nth term of an admissible sequence, if it exists, is always less than k_n (below, p. 454). This is equivalent to the following condition: there exists a k_0 such that $\langle n\rangle$ is admissible only if $n < k_0$, and for any admissible $\mathfrak{a}$ there exists a $k_\mathfrak{a}$ such that $\mathfrak{a}*\langle m\rangle$ is admissible only if $m < k_\mathfrak{a}$.[f] That is, at

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[b] See below, p. 453.

[c] In *1953*, p. 8, Brouwer admits a species of admissible sequences that is "not necessarily predeterminate". Although what this means is not altogether clear, he apparently does not intend to relax the requirement that we have stated. It seems, however, that Brouwer intends to allow the definition of a not necessarily predeterminate species to contain a free-choice parameter. Then, as Kreisel points out (*1964*, pp. 0.35–0.36), an example due to Kleene implies that the statement of the bar theorem given in *Brouwer 1953*, p. 14, requires modification.

[d] Brouwer's definition specifies a "sign series" (below, p. 453); this would suggest that species and free-choice sequences are not allowed. It seems that this restriction is not held to in subsequent intuitionistic writings. In *1952*, p. 142, Brouwer speaks of "*infinitely proceeding sequences* $p_1, p_2, \ldots$, whose terms are *chosen more or less freely* from mathematical entities *previously acquired*", which would seem to allow anything compatible with a step-by-step generation of mathematical entities. But see *Brouwer 1942*.

[e] The sequence is finite if an n is reached for which $\langle a_1, \ldots, a_n\rangle$ is terminal (that is, the second side of the alternative under (4) holds). Clearly, we cannot say of a given sequence that it either comes to an end or does not.

[f] We shall use lower-case German letters as variables ranging over finite sequences of natural numbers. $\mathfrak{a}*\mathfrak{b}$ is the concatenation of $\mathfrak{a}$ and $\mathfrak{b}$; if $\mathfrak{a} = \langle a_1, \ldots, a_n\rangle$ and $\mathfrak{b} = \langle b_1, \ldots, b_m\rangle$, then $\mathfrak{a}*\mathfrak{b} = \langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle$.

each stage there are only finitely many choices.

The *unit* continuum can be represented by a finitary spread if one sets a limit on how much smaller an interval can be than its predecessor (and on how small the initial interval may be), for then there are only finitely many choices.

Before we go on, the reader's attention is called to Brouwer's conception of a species. The definition that he gives in 1925 will be found below, p. 454. Later, species are defined as "properties supposable for mathematical entities previously acquired and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities that have been defined to be *equal* to it" (*1953*, p. 2). Since two species are equal if they have the same members, the notion of species has about the same role as that of class in nonintuitionistic mathematics.

Before we can explain the results of the paper below, we need to ask the question what meaning we can give to the notion of a *function* that maps one spread into another or into the natural numbers. It is by the analysis of this notion that Brouwer obtains the information necessary to prove the bar theorem and the uniform-continuity theorem.

The essence of the analysis is that, when a function is defined on a spread and has definite objects such as natural numbers as its values, its value for a given sequence that is an element of the spread must be determined by a finite number of terms of the sequence. If the value itself is to be a free-choice sequence, a certain initial segment of the argument must suffice to determine the first term of the value, a certain further segment to determine the second term, and so on.[g]

We must state this point with precision in order to avoid ambiguities. Consider a functional F that is defined on a spread M and whose values are natural numbers, as in the bar theorem. Then for every element α of M there exists a number n such that, if β agrees with α on the first n terms, that is, if the sequence

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[g] The first of these two statements is the continuity requirement that Brouwer makes in his proofs of the bar and fan theorems and of uniform continuity. But what in Theorem 1 below is stated to follow directly from the intuitionistic conception of a full function is a condition, weaker than the second statement, of *negative* continuity. It is puzzling why Brouwer states negative continuity with some fanfare and then goes on, at the beginning of § 2, to state quietly a stronger continuity requirement. The explanation seems to be as follows. Theorem 1 states the negative continuity of a full function in terms of the definition of § 1, that is, of the definition of negative continuity for a function of point *cores*. If f is a function that maps point cores onto point cores, it induces a function f_0 that maps *points* onto *points*. From the assumption of § 2 it follows immediately that, if f is a full function, f_0 is positively continuous. But it is not quite immediate that f is positively continuous.

Let ξ_0 be a point core; we must prove that f is continuous at ξ_0. Let p_0 be a point belonging to ξ_0; then the point $f_0(p_0)$ belongs to $f(\xi_0)$. Note that a point p is a sequence of intervals $p(n)$; we can denote by "$\overline{p}(m)$" the sequence of the first m intervals of p. From the continuity requirements of § 2 it follows that for any n we can find an m such that, if $\overline{p}(m) = \overline{p}_0(m)$, then $\overline{[f_0(p)]}(n) = \overline{[f_0(p_0)]}(n)$.

Given $\varepsilon > 0$, chose n_0 so that the diameter of $[f_0(p_0)](n_0) < \varepsilon$. For each m let a_m and b_m be the end points of $p_0(m)$. Let m_0 be the m obtained as above with $n = n_0$. Now let

$$\delta = \tfrac{1}{2}(\min(a_{m_0+1} - a_{m_0}, b_{m_0} - b_{m_0+1})).$$

δ is positive since $p_0(m_0 + 1)$ lies entirely within $p_0(m_0)$. Suppose that $|\xi - \xi_0| < \delta$ and that p is a point of ξ. Let k be such that the diameter of $p(k) < \delta$. Then $p(k)$ lies entirely within $p(m_0)$. Therefore p coincides with the point $q = p_0(0), \ldots, p_0(m_0), p(k), p(k+1), \ldots$, and $f_0(p)$ coincides with $f_0(q)$. Since $\overline{[f_0(q)]}(n_0) = \overline{[f_0(p_0)]}(n_0)$, and $f_0(q)$ is a point of $f(\xi)$, we have $|f(\xi) - f(\xi_0)| < \varepsilon$, q. e. d.

Since the number δ obtained for a given ξ_0 and a given ε depends on the particular point p_0 that represents ξ_0, the argument does not show that there is a *function* giving δ in terms of ξ_0 and ε. The existence of such a function is equivalent to uniform continuity in some neighborhood containing ξ_0 and presumably cannot be proved without the fan theorem.

of natural numbers of length n is the same for both, $F(\alpha) = F(\beta)$.

This implies that F can be represented by f, a function from finite sequences of natural numbers to natural numbers, in the following sense: if, in attempting to compute $F(\alpha)$ on the basis of the choices for α prescribed by the sequence $\mathfrak{a}$, we reach a point at which we lack the information about α needed to continue the computation, we set $f(\mathfrak{a}) = 0$. If we can complete the computation without meeting such an obstacle, we set $f(\mathfrak{a}) = F(\alpha) + 1$. This equation will hold for any α generated by a sequence of choices beginning with the choices of $\mathfrak{a}$.

The species μ_1 of § 2 in the text below can be identified with the species of admissible sequences $\mathfrak{a}$ for which $f(\mathfrak{a}) \neq 0$, but $f(\mathfrak{b}) = 0$ for any proper initial segment $\mathfrak{b}$ of $\mathfrak{a}$. Brouwer says that a sequence is *secured* if it belongs to μ_1, has an initial segment belonging to μ_1, or is inadmissible. The argument for the bar theorem, as well as for later results along the same lines, turns on an analysis of the species of *unsecured* sequences. If $\langle a_1, \ldots, a_n\rangle$ is secured and either $n = 1$ or $\langle a_1, \ldots, a_{n-1}\rangle$ is unsecured, $\langle a_1, \ldots, a_n\rangle$ is said to be *immediately secured.*

The important mathematical content of the paper is contained in the bar theorem. The fan theorem and the uniform-continuity theorem are corollaries. Below, the bar theorem is stated in the fourth paragraph of § 2. The claim is that, for a functional F from a spread M to the natural numbers, the species of unsecured sequences is capable of a certain kind of well-ordered construction. In order to explain this, we must make some remarks about Brouwer's theory of well-ordering.

The basic definitions are given below, pp. 456–457. I use the terminology presented in these definitions. It follows from the definition of a well-ordered species that with each well-ordered species S there is associated a species S' of finite sequences of natural numbers, the members of S' being the subscript sequences for the constructional underspecies of S. We can specify S, up to isomorphism, by giving S' and stating, for each sequence associated with a primitive species, whether the element of that species is a full or a null element.

Consider now the following ordering of finite sequences: $\langle a_1, \ldots, a_n\rangle \prec \langle b_1, \ldots, b_m\rangle$ if either

(1) $m < n$ and $a_i = b_i$ for $i = 1, \ldots, m$

or

(2) for some $i < \min(m, n)$, $a_j = b_j$ for all $j \leqq i$, while $a_{i+1} < b_{i+1}$;

that is, the sequences are so ordered by $\prec$ that an extension of a sequence *precedes* it and that otherwise two sequences are ordered lexicographically. Clearly, this is a primitive recursive linear ordering.[h] If $S_\mathfrak{a}$ and $S_\mathfrak{b}$ are the constructional underspecies of S with the subscript sequences $\mathfrak{a}$ and $\mathfrak{b}$ respectively, then $\mathfrak{a} \prec \mathfrak{b}$ if and only if $S_\mathfrak{a}$ precedes $S_\mathfrak{b}$ in the construction of S, that is, $S_\mathfrak{a}$ is a constructional underspecies of $S_\mathfrak{b}$, or every element of $S_\mathfrak{a}$ precedes every element of $S_\mathfrak{b}$ in the ordering of S. Since the latter side of the alternative must hold if $S_\mathfrak{a}$ and $S_\mathfrak{b}$ are primitive species, the ordering $\prec$ restricted to the subscript sequences of primitive species is isomorphic (as a linear ordering) to the ordering of S. The ordering $\prec$ restricted to S' satisfies the condition that every descending chain is finite, by virtue of the well-founded nature of the construction of S. Thus it is a well-ordering according to the classical conception.

The bar theorem can now be stated as follows. Let F be a functional that to each element α of a spread M assigns a natural number. With F is associated the species T of its unsecured and immediately secured sequences. Then T is the species S' of subscript sequences of

[h] We identify the sequence $\langle a_1, \ldots, a_n\rangle$ with the number $\prod_{i<n} p_i^{a_{i+1}+1}$.

some well-ordered species S. The elements of S can be taken to be the immediately secured sequences of F. The sequence $\mathfrak{a}$ is a full element if it is admissible (and a fortiori if $f(\mathfrak{a}) \neq 0$) and a null element otherwise.[1]

All this means that T can be inductively defined as follows:

(1) If $\mathfrak{a}$ is immediately secured, then $\mathfrak{a} \varepsilon T$;

(2) If $\mathfrak{a}*\langle n\rangle \varepsilon T$ for every n, then $\mathfrak{a} \varepsilon T$.

Hence we have the following *induction principle*: If a property holds of every immediately secured sequence and holds of $\mathfrak{a}$ if it holds of $\mathfrak{a}*\langle n\rangle$ for every n, then it holds of every sequence in T. In recent writings (*Spector 1961*, p. 9, *Kreisel 1963*) an essentially equivalent principle is formalized under the title "bar induction". Different versions of this principle are stated and compared in *Kleene and Vesley 1965*, § 6, where, however, the name "bar induction" is not used, and in *Howard and Kreisel 1966*.

An equivalent statement is that the representing function f of F belongs to a certain inductively generated species K of functions from finite sequences of natural numbers to natural numbers, defined as follows:

(1) Any constant function belongs to K;

(2) If H enumerates a sequence of functions in K, then

$$\lambda\langle a_0, \ldots, a_n\rangle\{H(a_0)\}(\langle a_1, \ldots, a_n\rangle) \varepsilon K.$$

Theorem 2, later called the fan theorem, states that, if the spread M on which F is defined is finitary, there can be found an n such that, for any α, the value of F depends only on the first n choices for α. This follows because one can show by an induction parallel to the generation of T that in this case there are only finitely many unsecured sequences. Clearly, the fan theorem in effect asserts the uniform continuity of F. Since, as we indicated above, the points of the unit continuum can be generated as a finitary spread, functions defined everywhere on the unit continuum, with points of the continuum as values, are uniformly continuous (Theorem 3).

The result of this analysis is that many definitions of functions of a real variable that, from the classical point of view, assign a value to the function for each value of the argument do not do so intuitionistically. The question arises whether one can single out certain subspecies of the continuum such that a function defined on such a subspecies will be analogous to a classically everywhere defined function. Such a function could be said to be *pseudo-full*. In the sections of the paper that are not reprinted here, Brouwer discusses a number of possible criteria for the domain D of a pseudo-full function of the unit continuum. Clearly, D should be a species that possesses a property classically equivalent to coincidence with the unit continuum. The one that Brouwer selects is *congruence*—that it is absurd that there should be a point of the unit continuum not coinciding with any point of the species (*1923d*, p. 255). The further requirement concerns measure: for *every* measure on the unit continuum D must be measurable and possess the measure 1.

A point that requires some discussion is the nature of the proof of the bar theorem. Even if we accept the continuity condition as expressing part of what we mean by a constructive functional on a spread, this is not sufficient for the proof. Brouwer goes on to exploit more fully than in any other intuitionistic argument the following peculiarity of intuitionistic mathematics: the supposition that a mathematical proposition is *true* is just the supposition that one has a (constructive) proof of it. In particular this will be the case, given a sequence $\mathfrak{a}$, for the statement that $\mathfrak{a}$ is *securable*, that

[1] This statement differs only in minor details from that of the text (fourth paragraph of § 2, p. 461 below).

is, that every sequence $\mathfrak{a}$, $\mathfrak{a}*\langle b_1\rangle, \ldots$, $\mathfrak{a}*\langle b_1, \ldots, b_m\rangle, \ldots$ contains a term that is secured. Brouwer goes on to make a controversial assumption about the possible form of a proof of such a statement. He claims that a proof of securability is based on the "givenness" of the secured sequences and on the relations between sequences that are formed by the composition of the relation of immediate succession, that is, the relation between $\mathfrak{a}$ and $\mathfrak{a}*\langle n\rangle$. Then the proof can be brought into a canonical form that uses only inferences resting upon the basic relations:

(1) If $\mathfrak{a}$ is (immediately) secured, it is securable;

(2) If $\mathfrak{a}$ is securable, so is $\mathfrak{a}*\langle n\rangle$ (ζ-inference);

(3) If $\mathfrak{a}*\langle n\rangle$ is securable for every n, $\mathfrak{a}$ is securable ($\digamma$-inference).

In the remainder of the argument it is shown that, for each unsecured sequence $\mathfrak{a}$ for which the canonical proof establishes that it is securable, there exists a well-ordered construction of the species $T_\mathfrak{a}$ of descendants of $\mathfrak{a}$ in T. It seems to follow, and indeed this is explicitly stated in *Brouwer 1953*, that the ζ-inferences are superfluous. We obtain a well-ordered construction in T itself by one more second generating operation (below, p. 456), taking as the nth term $T_{\langle n\rangle}$ if $\langle n\rangle$ is admissible and a species consisting of a single null element (below, p. 456) otherwise.

What can be said in justification of the claim that the proofs of securability have a canonical form? It does not seem to be at all evident. Indeed, it seems not to follow from Brouwer's remark that no other basis is available for the proofs than the relations of a sequence to those immediately issuing from it. For, perhaps, the proof might make some use of these relations that is not reducible to inferences directly involving the notion of securability.

Unfortunately, Brouwer's thesis that mathematics is independent of language and logic makes it difficult and perhaps impossible to consider possible counterexamples to his claim. However, it seems that his point of view implies the existence of *cut-free* canonical proofs, in which inductions are replaced by infinitary inferences (footnote 8 below, p. 460). This might be the answer to the objection that the assertion (represented as "q") of the securability of a sequence might be inferred from statements "p" and "$p \supset q$" while there might be no reason to expect the proofs of these premisses to contain the inferences that he claims. That modus ponens should be eliminable is suggested by Heyting's subsequent explanation of the conditional, together with the thesis that mathematics is independent of logic. According to Heyting "$p \supset q$" is the claim that there exists a method of reaching a proof of "q" from one of "p". Now, if the logical connectives are explained in terms of a notion of proof, one might expect that the "proofs" referred to in the explanation of a statement not containing the conditional should not themselves contain conditionals.

The general intuitionistic conceptions leave the meaning of quantification over free-choice sequences somewhat vague. It is made clearer by Brouwer's conception of a spread and by the continuity requirement, but there is room for further clarification. In the case of universal quantification over natural numbers, ordinary induction provides a very clear proof procedure, which arises directly from the inductive generation of the sequence of natural numbers itself. Nothing comparable is available for free-choice sequences. Indeed, it is not even certain whether they are to be regarded as individual mathematical objects at all or whether quantification over them is to be regarded as a *façon de parler*.

In the latter case we may be free to stipulate some criteria for the truth of statements containing such quantification. Even if free-choice sequences are individual mathematical objects, the notion seems vague enough, so that a stipulation as to truth or proof conditions of statements of certain forms may serve to clarify it. Indeed, Brouwer himself suggests in footnote 7, p. 460 below, that a stipulation underlies the bar theorem; he says that, "when carefully considered from the intuitionistic point of view", securability is just the property defined by an inductive definition like that given above for the species T.

In summary, Brouwer's justification for the bar theorem is certainly not evident or even satisfactory. His work makes clear that to obtain powerful results in intuitionistic analysis it is necessary to make some strong assumption that exploits the specifically intuitionistic force of quantification over free-choice sequences. If the assumptions that he makes are less evident than, say, the axiom of choice is in terms of the classical conception of set, this may be due to the newness of the whole subject.

Another point worth mentioning is that the well-ordering of unsecured sequences of continuous functionals was used in a classical context in descriptive set theory, prior to Brouwer's publication. Let S be a topological space, and suppose that we have a function that to each sequence $\mathfrak{a}$ assigns a subset $M_{\mathfrak{a}}$ of S. Then M is defined by the *operation A* applied to the sets $M_{\mathfrak{a}}$ if

$$M = \bigcup_{\alpha} \bigcap_{n=0}^{\infty} M_{\langle \alpha(0), \ldots, \alpha(n) \rangle},$$

that is, if

$$x \,\varepsilon\, M \equiv (\exists \alpha)(n)(x \,\varepsilon\, M_{\langle \alpha(0), \ldots, \alpha(n) \rangle})$$

(α ranging over one-place number-theoretic functions). Then we can say that a sequence $\mathfrak{a}$ is *secured* with respect to x if $x \notin M_{\mathfrak{a}}$; if α satisfies the condition[j]

$$\alpha(m) = \mathfrak{a}_m \text{ for } m < \tilde{\mathfrak{a}},$$

then in case $\mathfrak{a}$ is secured we have

$$(\exists n)(x \notin M_{\langle \alpha(0), \ldots, \alpha(n) \rangle}).$$

Hence, if every unsecured sequence is securable, we have

$$(\alpha)(\exists n)(x \notin M_{\langle \alpha(0), \ldots, \alpha(n) \rangle}),$$

that is, $x \notin M$. This will hold if and only if the ordering $\prec$ restricted to sequences unsecured with respect to x is a well-ordering. So we have

$$\bar{M} = \hat{x}(\prec \text{ well-orders } \hat{\mathfrak{a}}(x \,\varepsilon\, M_{\mathfrak{a}})).$$

Taking the real line as the set S, Suslin (*1917*) defines a class of sets called *A-sets*, which are those subsets of S that can be obtained by the operation A from systems of closed intervals. These sets are the same as the *analytic sets*, which can be defined in other ways, for example, as the images of S by functions continuous for all but countably many arguments or as the projections of Borel subsets of the plane. Suslin states that a set M is a Borel set if and only if both M and $\bar{M}$ are A-sets. A proof of this that uses the above definition of $\bar{M}$ in terms of unsecured sequences was given by Luzin and Sierpiński (*1918*, pp. 36–42). Extensive use of this method was made by Luzin in the twenties (*1927*; *1930*, chap. III), for example, to prove the first and second separation principles for analytic sets. Luzin (*1927*, pp. 2–3; *1930*, pp. 197–200) finds the germ of the idea in a construction by Lebesgue (*1905*).[k] From Brouwer's eminence as a topologist it would be plausible to conclude that he knew Luzin and Sierpiński's proof, but direct evidence is lacking.

The ideas of Brouwer and of the descriptive set theorists come together in Kleene's work on the hyperarithmetic and analytic hierarchies (*1955*). If we let the space S mentioned above be the set

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[j] $\mathfrak{a}_m$ = the $(m+1)$th term of $\mathfrak{a}$, $\tilde{\mathfrak{a}}$ = the least m for which $\mathfrak{a}_m$ does not exist.

[k] A special case of the operation A was introduced by Aleksandrov (*1916*). Although he does not give a definition of A-sets that is equivalent to Suslin's, he proves in effect that every Borel set is an A-set.

of natural numbers, the Σ_1^1 sets are just those that we obtain by applying the operation A, with the condition that "$x \varepsilon M_{\mathfrak{a}}$" be primitive recursive. In this case the ordering of unsecured sequences is also primitive recursive. Kleene seems to have derived the idea of the well-ordering from Brouwer by observing that in this case $(\alpha)(\exists n)(x \notin M_{\langle\alpha(0),\ldots,\alpha(n)\rangle})$ says that a certain recursive (and therefore continuous) functional is everywhere defined. He can thus show that any Π_1^1 set is recursive in the set O of notations for recursive ordinals. Spector (*1955*) shows that any Π_1^1 set is recursive also in the set W of Gödel numbers of recursive well-orderings. (Since W and O are Π_1^1, they are therefore recursive in each other.)

Further use by Kleene of the well-ordering (for example, to show that the hyperarithmetic sets are those that are Σ_1^1 and Π_1^1) is very close to that of descriptive set theory. The analogies are made very clear by Addison (*1958*), who, however, neglects to mention Kleene's debt to Brouwer.

As this example illustrates, the greatest influence of Brouwer's ideas has been on the development of theories of effectiveness and constructivity at higher types, in which the analysis of unsecured sequences is now a standard tool. Application of conceptions related to the bar theorem in proof theory has become quite extensive in recent work by Spector, Kreisel, and others. Although this work was motivated mainly by the consistency problem for classical analysis, it has also served to clarify the intuitionistic ideas themselves. The hope entertained by many that the idea of the bar theorem would yield a proof of the consistency of classical analysis led to a precise result by Spector (*1961*), who proved the consistency of classical analysis relative to that of a quantifier-free system containing functionals of arbitrary finite types and a schema for the definition of functionals by "bar recursion". But the form of "bar induction" that would serve to justify Spector's form of bar recursion is a generalization in which the "unsecured sequences" can be objects of any finite type. It is thus a substantial extension, for which no constructive foundation is known, of what has previously counted as intuitionistic mathematics.

Charles Parsons[1]

To help the reader in his study of the paper, we now print, in their logical order, the definitions, culled from Brouwer's writings, of a number of intuitionistic notions used in the paper.[m]

"A set ⟦*Menge*⟧ is a law on the basis of which, if repeated choices of arbitrary natural numbers ⟦Nummer⟧ are made, each of these choices either generates a definite sign series ⟦Zeichenreihe⟧, with or without termination of the process, or brings about the inhibition of the process together with the definitive annihilation of its result; for every $n > 1$, after every unterminated and uninhibited sequence of $n - 1$ choices, at least one natural number can be specified that, if selected as the nth number, does *not* bring about the inhibition of the process. Every sequence of sign series generated in this manner by an unlimited choice sequence ⟦unbegrenzten Wahlfolge⟧ (and hence generally not representable in a finished form) is called an *element of the set*. We shall also speak of the common mode of formation of the elements of a set M as, for short, *the set M*." (*Brouwer 1925*, pp. 244–245 (a footnote is omitted); see also *1918*, p. 3, and *1919b*, pp. 204–205, or in the reprint pp. 950–951.)

Subsequently, the terms that Brouwer uses for this notion are "spreiding" (*1947*, in Dutch) and "spread" (*1953*).

[1] I am indebted to Dirk van Dalen, Burton Dreben, J. J. de Iongh, Yiannis Moschovakis, and especially to Georg Kreisel and the editor, for their assistance and suggestions.

[m] I am grateful to Professor Richard E. Vesley for having helped me to understand a number of passages in Brouwer's writings.

"If for every n in ζ ⟦the sequence of natural numbers, $1, 2, 3, \ldots$⟧ a natural number k_n is determined such that the inhibition of the process takes place whenever a natural number lying in ζ above k_n is selected at the nth choice, then the set is said to be *finitary* ⟦*finit*⟧." (*Brouwer 1925*, p. 245.)

"Two set elements are said to be *equal* ⟦*gleich*⟧, or *identical* ⟦*identisch*⟧, if we are sure that for every n the nth choice generates the same sign series for the two elements.

"Two sets are said to be *equal*, or *identical*, if for every element of one set an equal element of the other can be specified.

"The set M is called a *subset* ⟦*Teilmenge*⟧ of the set N if for every element of M an equal element of N exists.

"Sets and elements of sets are called *mathematical entities*.

"By a *species* ⟦*Spezies*⟧ *of first order* we understand a property (defined in a conceptually complete form) that only a mathematical entity can possess, and, if it does, the entity is called an *element of the species of first order*. Sets constitute special cases of species of first order.

"Two species of first order are said to be *equal*, or *identical*, if for every element of one species an equal element of the other can be specified.

"By a *species of second order* we understand a property that only a mathematical entity or a species of first order can possess, and, if it does, the entity or the species is called an *element of the species of second order*.

"Two species of second order are said to be *equal*, or *identical*, if for every element of one species an equal element of the other can be specified.

"In an analogous manner we define *species of nth order*, as well as their equality, or identity, n representing an arbitrary element of A ⟦the set of natural numbers⟧.

"The species M is called a *subspecies* ⟦*Teilspezies*⟧ of the species N if for every element of M an equal element of N exists. If, moreover, it is possible to specify an element of N that cannot be equal to any element of M, then M is called a *proper subspecies* of N.

"Two set elements are said to be *distinct* ⟦*verschieden*⟧ if the impossibility of their equality has been established, that is, if we are sure that, in the course of their generation, their equality can never be proved.

"Two species are said to be *distinct* if the impossibility of their equality has been established." (*Brouwer 1925*, pp. 245–246.)

"A species of which any two elements can be recognized either to be equal or to be distinct is said to be *discrete*." (*Brouwer 1925*, p. 246.)

"The species that contains those elements that belong either to the species M or to the species N is called the *union* of M and N, and it is denoted by $\mathfrak{S}(M, N)$." (*Brouwer 1925*, p. 247.)

"Two species M and N are said to be *disjoint* ⟦*elementefremd*⟧ if they are distinct and it is impossible that there exist an element of M and an element of N that are identical with each other." (*Brouwer 1925*, p. 247.)

"If M' and M'' are disjoint subspecies of N and if $\mathfrak{S}(M', M'')$ and N are identical, then we say that N *splits* ⟦*zerlegt* ist⟧ into M' and M''; we call M' and M'' *conjugate splitting species of* N, and M', as well as M'', is called a *removable* ⟦*abtrennbare*⟧ *subspecies* of N." (*Brouwer 1925*, p. 247; see also *Brouwer 1918*, p. 4, *Heyting 1956*, p. 39, where, instead of "removable", "detachable" is used, and *Brouwer 1953*, p. 6.) If M' and M'' are removable subspecies of N, we can decide whether an arbitrary element of N belongs to M' or to M''.

"If between two species M and N there can be established a one-to-one relation, that is, a law that with every element of M associates an element of N in such a way that equal elements of N correspond to equal, and only to equal, elements of M and that every element of N is associated with an element of M, we write $M \sim N$ and say that M and N have the same *cardinality* ⟦*Mächtigkeit*⟧, or *cardinal number*, or are *equipollent* ⟦*gleichmächtig*⟧." (*Brouwer 1925*, p. 247.)

"The simplest example of an infinite set is the set A itself, and we denote its cardinal number by a. Species that possess that cardinal number are said to be *denumerably infinite* ⟦*abzählbar unendlich*⟧." (*Brouwer 1925*, p. 249; see also *Brouwer 1918*, pp. 6–7, and *Heyting 1956*, p. 39.)

"A species M ⟦of cardinal number m⟧ satisfying the formula $m \leqq a$ is said to be *denumerable* ⟦*abzählbar*⟧. In particular, it is said to be *numerable* ⟦*zählbar*⟧ if it can be mapped one-to-one onto a *removable* subspecies of A." (*Brouwer 1925*, p. 255; see also *Brouwer 1918*, p. 7, and *Heyting 1929* and *1956*, p. 40.)

"A species P is said to be *virtually ordered* ⟦*virtuell geordnet*⟧ if an asymmetric relation, which will be called the *ordering relation*, is defined for the elements of a subspecies of the species of pairs (a, b) of elements of P; we express this relation by '$a < b$', '*a before b*', '*a to the left of b*', '*a lower than b*', '$b > a$', '*b after a*', '*b to the right of a*', or '*b higher than a*', and, if we express the identity of two elements p and q of P by the formula '$p = q$', stipulate that it possess the following '*order properties*':

(1) The relations $r = s$, $r < s$, and $r > s$ are mutually exclusive;

(2) From $r = u$, $s = v$, and $r < s$ it follows that $u < v$;

(3) From the simultaneous absurdity ⟦Ungereimtheit⟧ of the relations $r > s$ and $r = s$ it follows that $r < s$;

(4) From the simultaneous absurdity of the relations $r > s$ and $r < s$ it follows that $r = s$;

(5) From $r < s$ and $s < t$ it follows that $r < t$." (*Brouwer 1925a*, p. 453; see also *Brouwer 1918*, p. 13, and *Heyting 1955*, p. 33, and *1956*, p. 107.)

"The virtually ordered species P is said to be *everywhere dense in the extended sense*, or, for short, *everywhere dense*, if between any two distinct elements of P there lie elements of P, and it is said to be *everywhere dense in the strict sense* if moreover an element of P can be specified and elements of P lie to the right as well as to the left of an arbitrary element of P." (*Brouwer 1925a*, p. 454; see also *Brouwer 1918*, p. 16.)

"If between two virtually ordered species P and Q there has been established a one-to-one relation that leaves the ordering relations invariant, we say that P and Q possess the same *ordinal number*, or are *similar*." (*Brouwer 1925a*, p. 455; see also *Brouwer 1918*, p. 14.)

"A virtually ordered species is said to be *ordered* if an ordering relation obtains for *every* pair (a, b) of *distinct* elements" (*Brouwer 1925a*, p. 455 (a footnote is omitted); see also *Heyting 1956*, p. 106.)

"A discrete ordered species is also said to be *completely ordered*." (*Brouwer 1925a*, p. 455.)

The ordinal number of the set A in its natural ordering is denoted by ω.

"Ordered species of ordinal number ω are also called *fundamental sequences* ⟦*Fundamentalreihen*⟧." (*Brouwer 1925a*, p. 455; see also *Brouwer 1918*, p. 14.)

"Let R be a virtually ordered species of virtually ordered species N such that equal elements e of $M = \mathfrak{S}(N)$ always belong only to equal species N and that equal species N are always virtually ordered in the same way. We denote the ordering relations of the given virtual orderings of R and the N by $>$ and $<$, and we define a virtual ordering of M as follows: We write $e' \gtrdot e''$, or $e'' \lessdot e'$, if either $N' > N''$ or both $N' = N''$ and $e' > e''$; we write $e' \underline{\gtrdot}\, e''$, or $e'' \underline{\lessdot}\, e'$, if $e' \lessdot e''$ is impossible; we write $e' > e''$ if $e' \underline{\gtrdot}\, e''$, and, moreover, $e' \neq e''$. ... We call the species M, once it is virtually ordered as above, (or its ordinal number m) the *ordinal sum* ⟦*ordnungsgemäße*

Summe]] of the species N (or of their ordinal numbers n), and we call the formation of this sum the *addition* of the N (or of the n). In case R possesses a finite complete ordinal number or the ordinal number ω, the sign $+$ is used in the ordinary way to denote addition." (*Brouwer 1925a*, p. 456–457.)

"Let M be a denumerably infinite ordered species that is everywhere dense in the strict sense and whose elements are denumerated by the fundamental sequence $g_1, g_2, g_3, \ldots$. We put $\mathfrak{S}(g_1, g_2, \ldots, g_\nu) = s_\nu$. In M we establish an 'intercalation partition' [['Einschaltungsteilung']]; that is, we generate in M, by an unlimited sequence of free choices, a left and a right subspecies (an arbitrary element of the left subspecies preceding an arbitrary element of the right one) in such a way that the left and the right subspecies are determined successively in $s_1, s_2, s_3, \ldots$, the procedure being such that only a single element g_{α_ν} of s_ν may remain excluded from these subspecies of s_ν and that for every ν the element $g_{\alpha_{\nu+1}}$, if it exists, is identical either with g_{α_ν} or with $g_{\nu+1}$. The species of the intercalation partitions t of M (*corresponding to arbitrary distinct denumerations of* M) that 'coincide' with a certain intercalation partition t_1 of M, in the sense that an element of the left subspecies of one partition never lies to the right of an element of the right subspecies of the other, is what we call an *intercalation element* of M, and t, as well as t_1, is a 'partition' of this intercalation element. We virtually order the species of the intercalation elements e of M by adopting the following stipulations: we write $e' \lessdot e''$ if we can specify a partition t' of e', a partition t'' of e'', and two elements g_σ and g_τ of M that belong to the right subspecies of t' and to the left subspecies of t'', respectively; we write $e' \underline{\lessdot} e''$ if $e' \gtrdot e''$ is impossible; we write $e' < e''$ if $e' \underline{\lessdot} e''$ and also $e' \neq e''$. The method already used several times above yields the result that these stipulations indeed entail the validity of the five ordering properties. We call the thus virtually ordered species of intercalation elements of M the *continuum over* M and denote it by $K(M)$." (*Brouwer 1925a*, p. 467.)

"The *well-ordered species* are ordered species that are defined on the basis of the following stipulations:

(1) An arbitrary element of a well-ordered species is either an *element of the first kind* and will be called a '*full element*' [['*Vollelement*']] or an *element of the second kind* and will be called a '*null element*' [['*Nullelement*']];

(2) A species with a unique element, once this element has been provided with either the predicate of being a full element or the predicate of being a null element, will be called a well-ordered species and, more particularly, a *primitive species* [[*Urspezies*]];

(3) From known well-ordered species further well-ordered species are derived through the *first generating operation*, which consists in the addition of a non-vanishing finite number of known well-ordered species, and through the *second generating operation*, which consists in the addition of a fundamental sequence of such species.

"Every well-ordered species that played a role in the construction of the well-ordered species F according to the preceding paragraph is called a *constructional underspecies* [[*konstruktive Unterspezies*]] of F. The constructional underspecies that played a role in the last generating operation of F are called the *constructional underspecies of first order* of F and are distinguished from one another by a subscript ν, hence are denoted by $F_1, F_2, \ldots, F_m$ or by $F_1, F_2, F_3, \ldots$. The constructional underspecies of the first order of an F_ν are called the *constructional underspecies of second order* of F and are denoted by $F_{\nu 1}, F_{\nu 2}, \ldots, F_{\nu m}$ or by $F_{\nu 1}, F_{\nu 2}, F_{\nu 3}, \ldots$. The constructional underspecies of the first order of a $F_{\nu_1 \cdots \nu_n}$ are called the *constructional underspecies of* $(n + 1)$*th order* of F and are denoted by $F_{\nu_1 \cdots \nu_n 1}$, $F_{\nu_1 \cdots \nu_n 2}, \ldots, F_{\nu_1 \cdots \nu_n m}$ or by $F_{\nu_1 \cdots \nu_n 1}$, $F_{\nu_1 \cdots \nu_n 2}$, $F_{\nu_1 \cdots \nu_n 3}, \ldots$ (F itself is taken as the *constructional underspecies of* 0*th order* of F). Thus every primitive species used in the construction of F turns out to be a *constructional underspecies of some finite order* of F (although for suitably chosen primitive species of F this order can, of course, increase beyond any bound). To see that, we need only use the *inductive method*, that is, observe that the property in question is satisfied for primitive species, that, if $\xi = \xi_1 + \xi_2 + \cdots + \xi_m$ on the basis of the first generating operation and $\xi' = \xi_1 + \xi_2 + \cdots + \xi_{m-1}$ on the basis of the first generating operation ($m \geqq 2$), the property in question, in case it holds for $\xi_1, \xi_2, \ldots, \xi_m$ as well as for ξ', also obtains for ξ, and, finally, that, if $\xi = \xi_1 + \xi_2 + \xi_3 + \ldots$ on the basis of the second generating operation, the property in question, in case it holds for every ξ_ν, also obtains for ξ.

"By means of the inductive method we see that for an arbitrary well-ordered species F the species of the subscript sequences of the elements, as well as the species of the subscript sequences of the constructional underspecies, forms a removable subspecies of the species of finite sequences of natural numbers; further, that for an arbitrary constructional underspecies $F_{\nu_1 \cdots \nu_n}$ of F the cardinal number of the $F_{\nu_1 \cdots \nu_n \mu}$ is known." (*Brouwer 1926*, pp. 451–452; see also *Brouwer 1918*, pp. 22–23, and *Heyting 1955*, pp. 33–34.)

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§ 1

Following § 5 of my paper "Zur Begründung der intuitionistischen Mathematik I" we define the *κ-intervals* and the *λ-intervals* in the "naturally ordered" species of finite binary fractions;[1] in particular, we shall say that these intervals are $\kappa^{(\nu)}$*-intervals* or $\lambda^{(\nu)}$*-intervals*, respectively, if their length is equal to $2^{-\nu}$.

By a *point of the linear continuum* we understand an unlimited sequence of λ-intervals (the "generating intervals" of the point) such that each of them is contained, in the strict sense, in the preceding one; their size, therefore, converges positively[2] to zero.

If in a set every choice that does not lead to the inhibition of the process generates a λ-interval, while each of these λ-intervals is contained, in the strict sense, in the λ-interval generated by the preceding choice, the set is called a *point set of the linear continuum*.

Let the species of those points p of the linear continuum that "coincide" with a certain point p_1 of the linear continuum (by which we mean that every generating interval of p completely or partially covers every generating interval of p_1) be called

[1] *Brouwer 1925*, p. 253. [["In the number continuum an interval with the end points $a \,.\, 2^{-n}$ and $(a+1)2^{-n}$ or with the end points $a \,.\, 2^{-n}$ and $(a+2)2^{-n}$ (where a is an arbitrary integer and n an arbitrary natural number) is called a κ-interval or a λ-interval, respectively."]]

[["Naturally ordered" means in order of increasing magnitude. Binary fractions are fractions of the form $a \,.\, 2^{-n}$, where a is an integer and n a natural number. Brouwer adds the word "finite" because he regards these fractions as finite sums of the form $b + \sum_{\nu=1}^{r} \alpha_\nu 2^{-\nu}$, where b is an integer, r is a natural number, α_ν, for $1 \leqq \nu \leqq r-1$, is equal to 0 or 1, and α_r is equal to 1.]]

[2] *Brouwer 1923b*, p. 6 [[above, p. 339]].

a *point core* ⟦*Punktkern*⟧ *of the linear continuum.* In what follows we shall denote the point cores of the linear continuum by y.

The points or point cores that are generated exclusively by λ-intervals partially covering the interval (0, 1) are called *points* or *point cores*, respectively, *of the unit continuum.* In what follows, the point cores of the unit continuum are denoted by x.

In a way similar to that followed in § 7 of my paper "Zur Begründung der intuitionistischen Mathematik II"[3] for the species of the intercalation elements of a denumerably infinite ordered species that is everywhere dense in the strict sense, the species of the point cores of the linear continuum or of the unit continuum can also be virtually ordered; once provided with this virtual ordering, this species is called the *linear continuum* or the *unit continuum*, respectively.

If the "naturally ordered" species of finite binary fractions between 0 and 1 is denoted by M, we shall say that the point core π of the unit continuum and the element e of $K(M)$[3] *coincide* if no element of the right, or left, subspecies of an intercalation partition of M that belongs to e can ever lie to the left, or right, respectively, of a generating interval of a point of π. These coincidence relations obviously determine a *similarity correspondence*[4] between the unit continuum and $K(M)$.

By a *real function*, or, for short, a *function*, $f(x)$ of x we understand a law that, with each of certain point cores of the unit continuum, which will be denoted by ξ and form the "domain of definition" of the function, associates one point core of the linear continuum, which will be denoted by $\eta = f(\xi)$.

A function $f(x)$ is said to be *negatively continuous for the value* ξ_0 if, for an arbitrary fundamental sequence $\xi_1, \xi_2, \ldots$ that converges positively to ξ_0, the fundamental sequence $f(\xi_1), f(\xi_2), \ldots$ converges negatively[2] to $f(\xi_0)$.

A function $f(x)$ is said to be *positively continuous for the value* ξ_0, or, for short, *continuous for the value* ξ_0, if for every positive rational ε a positive rational a_ε can be determined such that for $|\xi - \xi_0| < a_\varepsilon$ the inequality $|f(\xi) - f(\xi_0)| < \varepsilon$ holds.

A function that is negatively continuous or positively continuous for every ξ will be called, for short, a *negatively continuous* or a *continuous function*, respectively.

A function $f(x)$ is said to be *uniformly continuous* if for every positive rational ε a positive rational a_ε can be determined such that for $|\xi_2 - \xi_1| < a_\varepsilon$ the inequality $|f(\xi_2) - f(\xi_1)| < \varepsilon$ holds.[5]

A function $f(x)$ is said to be *discontinuous for the value* ξ_0 if a natural number n and a fundamental sequence $\xi_1, \xi_2, \ldots$ that converges positively[2] to ξ_0 can be specified such that $f(\xi_1), f(\xi_2), \ldots$ all differ from $f(\xi_0)$ by more than $1/n$.

[3] *Brouwer 1925a*, p. 467 ⟦above, p. 456⟧.

[4] See *Brouwer 1925a*, p. 455 ⟦above, p. 455⟧.

[5] It is only for the sake of simplicity that the definitions of continuity have been brought into the metric form above, of which they are independent so far as their content is concerned. To see that, we resort to the denumerably infinite, everywhere dense, ordered sets μ' and μ'', of ordinals $1 + \eta + 1$ and η, respectively, that generate the intercalation elements corresponding to the x and y, respectively; we denumerate μ' and μ'' by fundamental sequences $g_1', g_2', \ldots$ and $g_1'', g_2'', \ldots$, respectively, we denote $\mathfrak{S}(g_1', g_2', \ldots, g_\nu')$ and $\mathfrak{S}(g_1'', g_2'', \ldots, g_\nu'')$ by s_ν' and s_ν'', respectively, and we understand by an i_ν' or an i_ν'' a closed interval of μ' or μ'', respectively, whose end elements belong to s_ν' or s_ν'', respectively, but whose interior contains at most one element of s_ν' or s_ν'', respectively. On this basis, then, a *uniformly continuous function*, for example, is a function such that, given an arbitrary denumeration of μ' and an arbitrary denumeration of μ'', we can, for every natural number m, determine a natural number n such that, if ξ_1 and ξ_2 belong to the same i_n', $f(\xi_1)$ and $f(\xi_2)$ belong to the same i_m''.

A function that is discontinuous for any specific value belonging to its domain of definition is also said, for short, to be *discontinuous*.

A function $f(x)$ is said to be *full* ⟦*voll*⟧ if its domain of definition coincides with the unit continuum.

THEOREM 1. *Every full function is negatively continuous.*

Proof. Let $f(x)$ be a full function, let ξ_0 be an arbitrary point core x, and let ξ_1, $\xi_2, \ldots$ be a fundamental sequence of point cores x that converges positively to ξ_0. We now assume for the moment that there exist a natural number p and a fundamental sequence $p_1, p_2, \ldots$ of monotonically increasing natural numbers such that $|f(\xi_{p_\nu}) - f(\xi_0)| > 1/p$ for every ν, and we define a point core ξ_ω of the unit continuum by starting from an unlimited sequence $\digamma_1$ of generating intervals of a point belonging to ξ_0 and then constructing, by means of an unlimited sequence of choices of λ-intervals, a point $\digamma_2$ of the unit continuum in such a way that we temporarily choose, for every natural number n that we have already considered, the first n intervals identical with the first n intervals of $\digamma_1$ but reserve the right to determine, at any time after the first, second, ..., $(m-1)$th, and mth intervals have been chosen, the choice of all further intervals (that is, of the $(m+1)$th, $(m+2)$th, and so on) in such a way that either a point belonging to ξ_0 or one belonging to a certain ξ_{p_ν} is generated. Then the function $f(x)$ is not defined for the point core ξ_ω containing $\digamma_2$; this brings us to a contradiction, and our assumption has proved to be illegitimate. But this means that the function $f(x)$ is negatively continuous.

Theorem 1, which is an immediate consequence of the intuitionistic point of view and has since 1918 frequently been mentioned by me in lectures and conversations, suggests the conjecture that Theorem 3 below, which asserts much more, is valid; I did not, however, succeed in proving it until much later.[6] The object of the following two sections is to present this proof in as lucid as way as possible.

§ 2

Let M be an arbitrary set, let μ be the denumerably infinite set of finite (inhibited or uninhibited) choice sequences $F_{sn_1 \cdots n_r}$ upon which M is based (where s and the n_ν represent the natural numbers chosen one after the other for the choice sequence in question), and let a natural number β be associated with each element of M. Then there is distinguished in μ a removable numerable subset μ_1 of uninhibited finite choice sequences such that with an arbitrary element of μ_1 the same natural number β is associated for all elements of M issuing from μ_1, while furthermore a proof ⟦Beweisführung⟧[6a] h is given that makes it apparent, for an arbitrary uninhibited element of μ, that every uninhibited infinite choice sequence issuing from it possesses an ⟦initial⟧ segment belonging to μ_1. (For an uninhibited element of μ is to be taken as belonging to μ_1 if and only if for it—but for none of its proper segments—the decision with respect to β, according to the *algorithm* of the rule establishing the correspondence, is *not* postponed to further choices; it is of course by no means excluded here that we can afterward also specify elements of μ that neither belong

[6] See *Brouwer 1924* and *1924a* ⟦and also *1923*, pp. 3–5⟧.

[6a] ⟦In a paper published in English (*1953*) Brouwer uses the expression "mathematical argument".⟧

to μ_1 nor possess a segment belonging to μ_1 but have the property that the same natural number is associated with all elements of M that issue from such an element of μ.)

If we say that an element of μ is *secured* when it either is inhibited or possesses a (proper or improper) segment belonging to μ_1, then μ splits into a numerable set τ of secured and a numerable set σ of unsecured finite choice sequences, and the proof h shows that an arbitrary F_s is *securable*, that is, that every infinite choice sequence that issues from it and is uninhibited for M possesses a certain segment belonging to μ_1.[7] Let $h_{sn_1\cdots n_r}$ be a proof in which the securability of the element $F_{sn_1\cdots n_r}$ of σ is derived; then what this securability and this proof rest upon is, if we leave aside the fact that μ_1 and the inhibited choice sequences of μ are given, exclusively the relations, obtaining between the elements of μ, that are formed by the composition of [[welche sich zusammensetzen aus]] *elementary relations* e of the kind obtaining between two elements $F_{mm_1\cdots m_g}$ and $F_{mm_1\cdots m_g m_{g+1}}$, of which one is an [[immediate]] extension [[Verlängerung]][7a] of the other. Now, if the relations employed in any given proof can be decomposed into basic relations, its "canonical" form (that is, the one decomposed into elementary inferences[8]) employs only basic relations; in the case of the canonical form $k_{sn_1\cdots n_r}$ of the proof $h_{sn_1\cdots n_r}$ we can therefore ultimately infer the securability of $F_{sn_1\cdots n_r}$ exclusively from a combination of the species $S_{sn_1\cdots n_r}$, formed from the elementary relations e connecting $F_{sn_1\cdots n_r}$ to $F_{sn_1\cdots n_{r-1}}$ and to the $F_{sn_1\cdots n_r\nu}$, with a property previously derived from arbitrary elementary relations e and also from the fact that μ_1 and the inhibited choice sequences are given. For the last step of $k_{sn_1\cdots n_r}$ we therefore must previously have established the securability either of $F_{sn_1\cdots n_{r-1}}$ or of *all* $F_{sn_1\cdots n_r\nu}$.

If we now call the derivation of the securability of an $F_{mm_1\cdots m_g}$ from that of

[7] When carefully considered from the intuitionistic point of view [[Intuitionistisch durchdacht]], this securability is seen to be nothing but the property defined by the stipulation that it shall hold for every element of μ_1 and for every inhibited element of μ, and that it shall hold for an arbitrary $F_{sn_1\cdots n_r}$ as soon as it is satisfied, for every ν, for $F_{sn_1\cdots n_r\nu}$. This remark immediately implies the well-ordering property for an arbitrary $F_{sn_1\cdots n_r}$. The proof carried out in the text for the latter property, however, seems to me to be of interest nevertheless on account of the propositions contained in its elaboration.

[7a] [[In a paper published in English (*1953*) Brouwer uses the expression "immediate descendant".]]

[8] Just as, in general, well-ordered species are produced by means of the two generating operations from primitive species (*Brouwer 1926*, p. 451), so, in particular, mathematical proofs are produced by means of the two generating operations from null elements and elementary inferences that are immediately given in intuition (albeit subject to the restriction that there always occurs a last elementary inference). These *mental* mathematical proofs that in general contain infinitely many terms must not be confused with their linguistic accompaniments, which are finite and necessarily inadequate, hence do not belong to mathematics.

The preceding remark contains my main argument against the claims of Hilbert's metamathematics. A second argument is that the way in which Hilbert seeks to settle the question (which, incidentally, was taken over from intuitionism) of the reliability of the principle of excluded middle is a vicious circle; for, if we wish to provide a foundation for the correctness of this principle by means of the proof of its consistency, this implicitly presupposes the principle of the reciprocity of the complementary species and hence the principle of excluded middle itself (see *Brouwer 1923c*, p. 252) [[Concerning this passage Brouwer (*1953*, p. 14, footnote 1) writes: "The equivalence of the principles of the excluded third and of reciprocity of complementarity, mentioned there in a footnote by way of remark, subsequently has been recognized as nonexistent. In fact, as was also shown in the present paper, the fields of validity of these two principles have turned out to be essentially different".]].

$F_{mm_1\cdots m_{g-1}}$ a *ζ-inference* and the derivation of the securability of an $F_{mm_1\cdots m_g}$ from that of *all* $F_{mm_1\cdots m_g\nu}$ a *ϝ-inference*, then the proof $k_{sn_1\cdots n_r}$ forms a well-ordered species of which every full element is formed by an elementary inference that, in case it constitutes the derivation of the securability of an element of σ, represents either a ϝ-inference or a ζ-inference.

We now assert that every element $F_{sn_1\cdots n_r}$ of σ possesses the *well-ordering property* [[*Wohlordnungseigenschaft*]], that is, that the subset $M_{sn_1\cdots n_r}$ of M determined by $F_{sn_1\cdots n_r}$ splits into a species of subsets M_α that is similar to the species of the full elements of a well-ordered species $T_{sn_1\cdots n_r}$, each of these subsets being determined by a finite initial segment F_α of choices that contains $F_{sn_1\cdots n_r}$ and belongs to μ_1. The species $T_{sn_1\cdots n_r}$ is constructed by means of generating operations w of the second kind, of which each corresponds to the inversion of the continuation, by a new free choice, of a certain finite initial segment of choices that is uninhibited for M. Then to a new choice that is inhibited for M or terminates an element of μ_1 there corresponds, for the operation w in question, a primitive species consisting of a null element or a full element, respectively.

For the proof of this assertion we denote by $f_{sn_1\cdots n_r}$ the species of those elements of σ whose securability we ascertain in the course of $k_{sn_1\cdots n_r}$, and we say that a constructional underspecies u of $k_{sn_1\cdots n_r}$ possesses the well-ordering property if every element of σ whose securability we ascertain in the course of u possesses the property of being well-ordered. Further we shall say that the *preservation property* [[*Erhaltungseigenschaft*]] holds for a constructional underspecies u of $k_{sn_1\cdots n_r}$ if, *in case* every element of $f_{sn_1\cdots n_r}$ upon whose securability the proof u is based possesses the well-ordering property, every element of $f_{sn_1\cdots n_r}$ whose securability is derived in the course of u likewise possesses the well-ordering property. Then, as we observe the generation of $k_{sn_1\cdots n_r}$, we see by means of the inductive method that for every constructional underspecies of $k_{sn_1\cdots n_r}$, hence in particular for $k_{sn_1\cdots n_r}$ itself, the preservation property holds. But from the preservation property for $k_{sn_1\cdots n_r}$ the well-ordering property immediately follows for $k_{sn_1\cdots n_r}$, hence for $F_{sn_1\cdots n_r}$.[9]

In case M is a *finitary* set, the well-ordered species $T_{sn_1\cdots n_r}$ has the same content[9a] as a well-ordered species $Q_{sn_1\cdots n_r}$ that is constructed without the use of null elements and, moreover, in a way parallel to the construction, discussed above, of $T_{sn_1\cdots n_r}$ namely, in a way such that to each operation w_α used for the construction of $T_{sn_1\cdots n_r}$

[9] If the securability of $F_{sn_1\cdots n_r}$ is ascertained in several proofs $k_{sn_1\cdots n_r}$ or in several places of one and the same proof $k_{sn_1\cdots n_r}$, the corresponding $T_{sn_1\cdots n_r}$ all are generation-equivalent, as follows by virtue of the inductive method when we observe the generation of one of them. This remark, incidentally, is superfluous for the proof above. [["Two well-ordered primitive species F' and F'' have *the same generating value* [[besitzen *denselben Erzeugungswert*]], or are said to be *generation-equivalent* [[*erzeugungsgleich*]], if the single element of which each of them consists is either a full element for both or a null element for both. Two well-ordered species F' and F'' are said to be *generation-equivalent* if for an arbitrary ν the two constructional underspecies of the first order F'_ν and F''_ν either both fail to exist or both exist and are generation-equivalent." (*Brouwer 1926*, p. 452.)]]

[9a] [["Two well-ordered species (or subspecies of well-ordered species) F' and F'' are said to *have the same content* [[heißen *inhaltsgleich*]] if the species of the full elements of F' and the species of the full elements of F'' are similar." (*Brouwer 1926*, p. 453.)]]

there corresponds a generating operation v_α of the first kind used for the construction of $Q_{sn_1\cdots n_r}$, the terms of v_α being similar, in sequence, to the species of the full elements of those terms of w_α that contain full elements. The well-ordered species $Q_{sn_1\cdots n_r}$ is therefore constructed by the exclusive use of generating operations of the first kind. But from this it follows that the species of elements of $Q_{sn_1\cdots n_r}$, as well as the species of full elements of $T_{sn_1\cdots n_r}$, is *finite*, hence that in particular for every natural number s the species of the full elements of T_s is finite. Thus a natural number z can be specified such that an arbitrary element of μ_1 possesses at most z subscripts; therefore the natural number β_e associated with an arbitrary element e of M is completely determined by the first z generating choices of e, and we have established the property expressed in the following

THEOREM 2. *If with each element e of a finitary set M a natural number β_e is associated, a natural number z can be specified such that β_e is completely determined by the first z choices generating e.*

§ 3

In the unit continuum we now determine for every natural number ν the k_ν-intervals $k'_\nu, k''_\nu, \ldots, k_\nu^{(s_\nu)}$, that is, the $\lambda^{(4\nu+1)}$-intervals, ordered from left to right, that partially cover the interval (0, 1). Then the finitary point set J formed by the nestings of intervals $k_1^{(\mu_1)}, k_2^{(\mu_2)}, k_3^{(\mu_3)}, \ldots$ (where each interval is contained, in the strict sense, in the preceding one) coincides with the species of the x; that is, every such nesting of intervals belongs to an x, and every x contains such a nesting of intervals.[10]

Now in the case of a full function $f(x)$ a nesting of λ-intervals $\lambda_1, \lambda_2, \ldots$ is associated with every nesting of intervals $k_1^{(\mu_1)}, k_2^{(\mu_2)}, \ldots$, and by Theorem 2 there exists, for every natural number ν, a natural number m_ν (of which we may assume that it does not decrease with increasing ν) such that λ_ν is determined by the choice of $k_1^{(\mu_1)}, k_2^{(\mu_2)}, \ldots, k_{m_\nu}^{(\mu_{m_\nu})}$. Hence for each ν only a finite number l_ν of λ-intervals can occur as λ_ν, and there exists for them a maximal width b_ν that converges to zero as ν increases beyond all bounds.

Let us denote by $t_\nu^{(\rho)}$ the interval that is concentric with $k_\nu^{(\rho)}$ and whose width is $\frac{3}{4}$ of the width of $k_\nu^{(\rho)}$, and let P_1 and P_2 be two arbitrary point cores of the unit continuum that are $\lessdot 2^{-4\nu-3}$, that is, $\lessdot \frac{1}{4}$ of the width of the k_ν-intervals, apart. Then a $t_\nu^{(\mu_\nu)}$ can be determined that contains both P_1 and P_2, and by means of this $t_\nu^{(\mu_\nu)}$ a nesting of intervals $k_1^{(\mu_1)}, \ldots, k_\nu^{(\mu_\nu)}, k_{\nu+1}^{(\sigma_1)}, k_{\nu+2}^{(\sigma_2)}, \ldots$ belonging to P_1 and a nesting of intervals $k_1^{(\mu_1)}, \ldots, k_\nu^{(\mu_\nu)}, k_{\nu+1}^{(\tau_1)}, k_{\nu+2}^{(\tau_2)}, \ldots$ belonging to P_2 can be determined.

Let ε be an arbitrary positive quantity that is positively different[10a] from zero. If we choose ν_ε so great that $b_{\nu_\varepsilon} < \varepsilon$ and if we put $2^{-4m_{\nu_\varepsilon}-3} = a_\varepsilon$, then, according to the

[10] If in an analogous way we consider a suitable *finitary set of pairs of points* that coincides with the species of the *pairs of point cores* of the unit continuum, then on the basis of Theorem 2 the *impossibility of splitting* ⟦*Unzerlegbarkeit*⟧ *the continuum* readily follows, that is, the property that, for an arbitrary splitting of the unit continuum into a discrete species of subspecies, one of these subspecies is identical with the unit continuum.

[10a] ⟦On this notion see *Brouwer 1919*, p. 3, lines 7u–5u, and *1923d*, p. 254, lines 3–5; compare Definition 1 in 2.2.3 of *Heyting 1956*, p. 19.⟧.

second paragraph of the present section, to any two elements of J for which μ_1, μ_2, $\dots$, $\mu_{m_{\nu_\varepsilon}}$ are equal there correspond two "values" of $f(x)$ whose difference is less than b_{ν_ε}, hence less than ε, in absolute value. According to the third paragraph of the present section, therefore, it is also the case that to any two point cores P_1 and P_2 of the unit continuum that are $\lessdot a_\varepsilon$ apart there correspond two values of $f(x)$ whose difference is less than ε in absolute value, so that $f(x)$ turns out to be *uniformly continuous* and we have proved

THEOREM 3. *Every full function is uniformly continuous.*