CS3110 Spring 2017 Lecture 20: Brouwer's Fan Theorem

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	Date for	Due Date
PS6	Out on April 24	May 8 (day of last lecture)

1 Brouwer's Fan Theorem and König's Lemma

Brouwer published his fan theorem in 1927 [1]. We have included an English translation of this famous article in the course resources. König proved his classical theorem in the same year [2]. The historical resources we know do not report whether there was a connection between Brouwer and König.

We can prove a classical version of Brouwer's Fan Theorem using König's Lemma. We can also prove the Fan theorem constructively, using Russian constructive mathematics which allows proving termination of programs by contradiction. Then we can use Brouwer's *Continuity Principle for Numbers* and the Fan theorem to prove that all real functions defined on a closed interval are uniformly continuous. We will look at some of these ideas in this lecture to explore the computational meaning of the Continuity Principle.

Here are informal statements of both König's Lemma and Brouwer's Fan Theorem (1927)

König's Lemma: Every finitely generated tree with an unbounded number of nodes has an unbounded path.

Definition: The *universal spread* consists of all unbounded sequences of natural numbers.

We don't think of unbounded sequences as completed, infinite sequences, but as growing sequences that can always be extended. A sequence could be recursively generated by a *law* that determines the next member of the sequence as a function of what has already been determined. Or, a sequence could be *freely chosen*, for instance, randomly chosen. In general, sequences can be determined partly by laws and partly by free choices.

Definition: A *fan* is a spread in which each choice comes from a finite list of numbers. A *path* in the fan is an unbounded sequence where each choice made is one of the numbers allowed by the fan.

Brouwer's Fan Theorem: Given a fan F and an effective assignment of an integer $\beta(p)$ to each path p in F, we can compute a uniform bound b in $\mathbb{N}$ such that for every path p in F, $\beta(p)$ is determined by the first b members of the sequence p.

1.1 The Fan Theorem informally

Dr. Bickford has proven results like this in Nuprl, and he helped me frame this very succinct and precise account of the Fan Theorem given below. Here is what Brouwer said about the theorem in his 1927 article:

"If with each element e of a finite spread M a natural number β_e is associated, a natural number z can be specified such that β_e is completely determined by the first z choices generating e.

The claim is that β_e does not depend on any unbounded choice sequence, only on finitely many elements in the underlying tree.

Here is a precise statement of the *Fan Theorem* using modern constructive type theory. First we need to specify the finite number of choices. It turns out that if we can prove the theorem when there are just two possibilities, then we can prove the theorem for any finite number of choices, so we can use the type $\mathbb{B}$ for the possible choices. Then a function of type $\mathbb{N}_n \rightarrow \mathbb{B}$ represents a finite sequence of choices, where $\mathbb{N}_n$ is the set $\{0, 1, \ldots, n-1\}$.

The paths in this fan are the "sequences" of type

$$\mathcal{C} = \mathbb{N} \ \rightarrow \ \mathbb{B}$$

as long as we allow free choice sequences (as well as lawlike sequences) in this type. Notice that if $f \in \mathcal{C}$ then f also has type $\mathbb{N}_n \rightarrow \mathbb{B}$.

Now, an effective association of an integer to each path is a function

$$\beta \in \mathcal{C} \rightarrow \mathbb{Z}$$

Brouwer's *Continuity Principle for Numbers* says that given such a β and a path p, then there is a number n such that $\beta(p)$ is known (i.e. computed, or "output") once the first n members of the sequence p have been chosen. And there is an effective test P_β that tells us whether we know $\beta(p)$.

This effective test P_β has type $n\colon \mathbb{N} \to (\mathbb{N}_n \to \mathbb{B}) \to \mathbb{B}$. And $P_\beta(n,p) = true$ when the first n members of p are enough to determine $\beta(p)$. We will write $P_\beta(n,p)$ when we really mean that $P_\beta(n,p) = true$.

Given all of that, the Fan theorem then follows from:

$$(\forall p : \mathcal{C}.\ \exists n : \mathbb{N}.\ P_\beta(n,p)) \Rightarrow \exists k : \mathbb{N}.\ \forall p : \mathcal{C}.\ \exists n : \mathbb{N}_k.\ P_\beta(n,p)$$

This says that if every path "hits the bar" P_β (i.e. $\beta(p)$ is determined at some finite n), then we can find a uniform bound k such that for all the (uncountably many) paths p in $\mathcal{C}$, $\beta(p)$ is determined by the first k members of the path.

We can compute this k from the test P_β by letting $p_t = \lambda x.true$ and $extend(n,s,b) = \lambda x.$ if $x < n$ then $s(x)$ else b and

$$k \ = \ Fan(P_\beta, 0, p_t)$$

Where the computable Fan function is

$$\begin{aligned} Fan(P,n,s) \quad = \quad & \text{if } P(n,s) \text{ then } n \text{ else} \\ & \text{let } k1 = Fan(P, n+1, extend(n,s,true)) \text{ in} \\ & \text{let } k2 = Fan(P, n+1, extend(n,s,false)) \text{ in} \\ & max(k1,k2) \end{aligned}$$

We now prove by contradiction that $Fan(P_\beta, 0, p_t)$ converges. We can do this using König's Lemma.

References

[1] L.E.J. Brouwer. Über definitionsbereiche von funktionen. *Mathematische Annalen*, 97:60–75, 1927.

[2] D. Konig. Über eine schlussweise aus dem endlichen ins unendliche. 3(2):121–130, 12 1927.