Constructive Synthetic Geometry

Ariel Kellison

April 11, 2017

Constructive Euclidean Geometry

- Our geometry is *constructive* in the sense that when we are required to prove the existence of a geometric object, we must develop an algorithm to construct the object.
- Significance to CS: the construction, or method of finding, a geometric object is equivalent to an algorithm.
- Many of the theorems (referred to as propositions) in Euclid's *Elements* are considered "constructive" in the sense that they provide a method for constructing a geometric object (e.g. a point of intersection). The tools for Euclid's constructions are the straightedge and compass.

Synthetic Geometry

- Synthetic systems of geometry are coordinate free and have the advantage of producing more general constructions (algorithms) than analytic systems.

Example

The disadvantage of generality: Let a and b be distinct points, and let L be the line through a and b. Let c and d be points on opposite sides of L, and let M be the line through c and d. Let e be the intersection of L and M.

1. Is e necessarily between c and d?
2. Is e necessarily between a and b?

- Synthetic systems can be too general; the axioms of the system supplement our geometric intuition by providing justifications for our decisions.

Axiomatic Systems

Axiomatic systems are composed of

Primitives (undefined terms)

Axioms (statements about the primitives)

Laws of Logic (we use Nuprl, which is utilizes *constructive* logic)

Theorems (logical consequences of the axioms)

Foundations of Axiomatic Systems

In our system:

Primitives (undefined terms)

- Objects : Points
- Relations (on points) :
  1. separation of two points (binary relation) $a \neq b$
  2. a point a separated from a line bc (ternary relation) $a \neq bc$
  3. non-strict betweeness (ternary relation) a_b_c
  4. congruence (quaternary relation) $ab = cd$

Some of Our Constructive Axioms

1. Separation of Points
   - Symmetry : $a \neq b \rightarrow \; b \neq a$
   - "Co-transitivity": $a \neq b \rightarrow \; \forall c \; [a \neq c \; \vee \; b \neq c]$
   - Reflexivity : $\sim (a \neq a)$
2. Separation of a point from a line ("positive triangle")
   - Symmetry : $a \neq bc \; \rightarrow c \neq ab \; \rightarrow b \neq ac$
   - Symmetry : $a \neq bc \; \rightarrow a \neq cb$
   - Relation to collinearity : $\sim (a \neq bc) \; \rightarrow collinear(abc)$
3. Extension $a, b, c, d : point \; \exists x : point[a\_b\_x \wedge bx = cd]]$
4. Circle Circle Intersection (continuity)
   $\forall a, b, c, d : point \; \exists u, v : point[au = ab \wedge av = ab \wedge cu = cd \wedge cv = cd]$
5. Line Circle Intersection (continuity)
   $\forall a, b, c, d : point \; \exists u, v : point[cd = cu \wedge cd = cv \wedge collinear(abuv)]$

Euclid's Axioms

1. First Axiom: Things which are equal to the same thing are also equal to one another.
2. Second Axiom: If equals are added to equals, the whole are equal.
3. Third Axiom: If equals be subtracted from equals, the remainders are equal.
4. Fourth Axiom: Things which coincide with one another are equal to one another.
5. Fifth Axiom: The whole is greater than the part.
6. First Postulate: To draw a line from any point to any point.
7. Second Postulate: To produce a finite straight line continuously in a straight line.
8. Third Postulate: To describe a circle with any center and distance.
9. Fourth Postulate: That all right angles are equal to one another.
10. Fifth Postulate: The parallel postulate...

A Constructive Account of Euclid's Proposition 9

Ariel Kellison

Spring 2017

1 Euclids' Proposition 9

To bisect a given rectilineal angle

Let the angle abc be the given rectilineal angle.
Thus it is required to bisect it.

I. Let a point d be taken at random on ab;

II. let be be cut off from bc equal to ad;

III. let de be joined;

IV. and on de let the equilateral triangle def be constructed;

V. let bf be joined;

VI. I say that the angle abc has been bisected by the straight line bf.

VII. For, since bd is equal to be and bf is common,

VIII. the two sides db, bf are equal to the two sides eb, bf respectively.

IX. And the base df is equal to the base ef;

X. therefore the angle dbf is equal to the angle ebf.

XI. Therefore the given rectilineal angle abc has been bisected by the straight line bf [1].

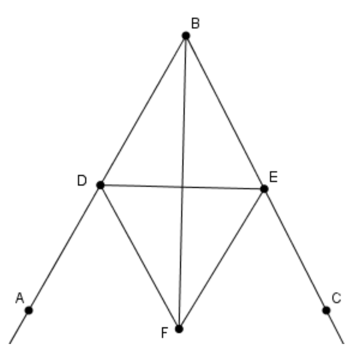

Figure 1: Euclid's Proposition 9.

2 Angles

2.1 A Formal Definition for Angles

To begin a constructive analysis of Euclid's proposition 9, we must first consider his definition of a plane rectilineal angle:

> *A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*
>
> *And when the lines containing the angle are straight, the angle is called rectilineal* [1].

One interpretation of Euclid's definition for a plane rectilineal angle could take the logical form

$$\angle a, b, c \leftrightarrow \ a \neq \ b \ \wedge \ a \neq \ c \ \wedge \ b \neq \ c \ \wedge \neg \mathbf{B}(abc) \ \wedge \neg \mathbf{B}(bca) \ \wedge \ \neg \mathbf{B}(cab),$$

where $\mathbf{B}(xyz)$ represents the primitive geometrical notion of strict betweeness and $\neq$ represents the distinctness of two points and is defined in terms of betweeness : $a \neq b \rightarrow \exists x[\mathbf{B}(axb)]$. Unfortunately, in our geometry, the definition of a plane rectilineal angle given above is not sufficient. Constructively, we require that at least one point can be found to witness that the two sides containing the angle do not coincide [2]. This is equivalent to our primitive notion of a point separated from a line, first introduced by Heyting in his doctoral thesis [3]. We communicate that the point a is separated from the line bc with the notation $a \neq bc$:

$$a \neq bc \rightarrow c \neq ba \ \wedge \ c \neq ab \ \wedge \ a \neq c \wedge \ \sim a\_b\_v \ \wedge \ \forall z \ [z \neq b \rightarrow collinear(abz) \rightarrow z \neq bc].$$

The constructive definition of a plane rectilineal angle is then

$$\angle a, b, c \leftrightarrow a \neq bc$$

Both definitions exclude straight angles (by not allowing collinearity between the three point), remaining faithful to Euclid's informal definition of a plane rectilineal angle.

2.2 Congruent Angles

We define congruent angles using the primitive quaternary relation of congruence on points (=):

$$\angle abc = \angle xyz \leftrightarrow a \neq b \ \wedge \ c \neq b \ \wedge \ x \neq y \ \wedge \ z \neq y$$
$$\wedge \ \exists a', c', x', z'[\mathbf{B}(baa') \ \wedge \ \mathbf{B}(bcc') \ \wedge \ \mathbf{B}(yxx') \ \wedge \ \mathbf{B}(yzz') \ \wedge \ ba' = yz' \wedge a'c' = x'z'].$$

Congruent angles are therefore contained by congruent side lengths and have the sides opposite them congruent (SSS triangle congruence).

3 A Constructive Account of Euclid's Proposition 9

After stating proposition 9 formally as

$$\forall a, b, c \ [a \neq bc] \Rightarrow \exists f \ [\angle abf = \angle cbf]$$

We proceed with Euclid's construction with a constructive account in mind.

3.1 Lines I & II

The diagram traditionally attributed to Proposition 9, shown in Figure 1, infers that the point d is chosen such that $\mathbf{B}(bda) \wedge (|bd| < |bc| \rightarrow \mathbf{B}(bec))$. The text of the proof, however, infers that d is chosen such that $\mathbf{B}(bda) \ \wedge \ \neg(\neg \mathbf{B}(bec) \ \wedge \ \neg \mathbf{B}(bce))$.

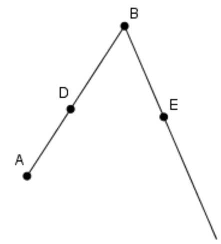

Figure 2: The position of c relative to e is unknown; both lie on the same side of b, i.e. $\neg(\neg\mathbf{B}(bec) \wedge \neg\mathbf{B}(bce))$

3.2 Line III

That de constitutes a line relies on the distinctness of d and e, following from $a \neq bc$: d is collinear with and distinct from a and b and e is collinear with and distinct from c and b.

3.3 Line IV

The construction of an equilateral triangle on the segment de does not guarantee that the vertex of the equilateral triangle, f, will be on the opposite side of de from $\angle b$. Furthermore, if $\measuredangle b = 60°$ then Δdef is equilateral, f and b potentially coincide, and bf does not determine a line. Thus a constructive proof of proposition 9 can not rely on Proposition 1 [2].

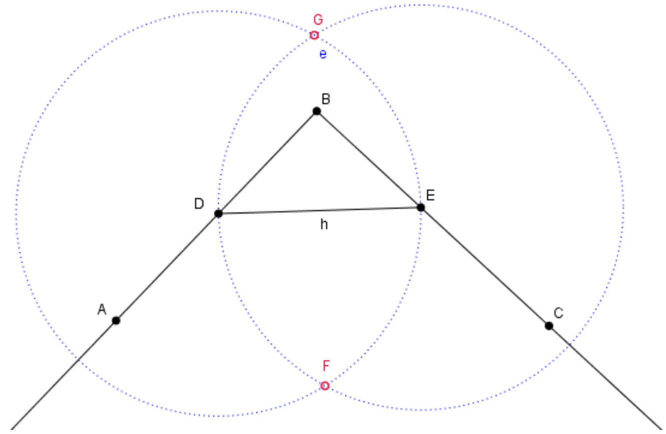

Figure 3: An equilateral triangle constructed on de using Proposition 1 could result in the construction of either point f or g.

Instead of relying on Proposition 1, the proof can be carried out constructively by constructing the midpoint of the base of the isosceles triangle Δdbe. In Euclid's Book I, the midpoint of a segment is not constructed until Proposition 10. Proposition 10 constructs the midpoint for *any* segment and we wish to construct the midpoint for a *a specific segment*, namely the segment that is the base of an isosceles triangle. For this we use a version of Tarski's theorem 7.25 [4]:

$$\forall a, b, c\ [ab = bc] \Rightarrow \exists x[\mathbf{T}(axc)\ \wedge\ ax = xc]\ [4].$$

In our constructive geometry, the theorem is stated using point line separation:

$$\forall a, b, c[a \neq bc\ \wedge ab = bc] \Rightarrow \exists x[\mathbf{B}(axc)\ \wedge\ ax = xc].$$

The proof of this lemma relies on elementary lemmas for the primitive relations of congruence and betweeness, and the inner pasch and five segment axioms.

Using Tarksi's lemma 7.25 allows us to construct f as the midpoint of de without the axiom of circle-circle continuity.

3.4 Line V

That bf constitutes a line relies on the distinctness of b and f, $b \neq f$, which follows from $a \neq bc$, as f is strictly between d and e, d is collinear with b and a, and e is collinear with b and c.

3.5 Line VI to XI

To prove the bisection of $\angle abc$, Euclid relies on proposition 8, SSS congruence for triangles, to show that $\angle dbf = \angle ebf$. To bisect the angle constructively, it would be necessary to show that $\angle dbf = \angle ebf$ is enough evidence to prove $\angle abf = \angle cbf$. We infer only $\neg(\neg\mathbf{B}(bec) \ \wedge \ \neg\mathbf{B}(bce))$ as the relationship between e and c for our construction, which makes it non-trivial to show that $(\angle dbf = \angle ebf) \rightarrow (\angle abf = \angle cbf)$ and so congruence between $\angle abf$ and $\angle cbf$ is instead proven directly.

References

[1] Thomas L. Heath Euclid and Dana Densmore. *Euclid's Elements: All Thirteen Books Complete in One Volume : the Thomas L. Heath Translation*. Green Lion Press, 2002.

[2] Michael Beeson. Brouwer and euclid. 2016.

[3] Jan von Plato. Proofs and types in constructive geometry (tutorial). Technical report, Rome, Italy, 2003.

[4] W.Szmielew W.Schwabhäuser and A. Tarski. *Metamathematische Methoden in der Geometrie*. Springer-Verlag, 1983.