is countably infinite. A similar proof using (1.2) shows that $\mathbb{Z} \times \mathbb{Z}$ is countably infinite

A set which is in one-one correspondence with $\mathbb{Z}_n$ is said to *have n elements*, and to be *finite*. Every finite set is countable.

It is not true that every countable set is either countably infinite or subfinite. For example, let A consist of all positive integers n such that both n and $n+2$ are prime; then A is countable, but we do not know if it is either countably infinite or subfinite. This does not rule out the possibility that at some time in the future A will have become countably infinite or subfinite; it is possible that tomorrow someone will show that A is subfinite. This set A has the property that if it is subfinite, then it is finite. Not all sets have this property.

2. The Real Number System

The following definition is basic to everything that follows.

(2.1) **Definition.** A sequence (x_n) of rational numbers is *regular* if

(2.1.1) $$|x_m - x_n| \leqq m^{-1} + n^{-1} \quad (m, n \in \mathbb{Z}^+).$$

A *real number* is a regular sequence of rational numbers. Two real numbers $x \equiv (x_n)$ and $y \equiv (y_n)$ are *equal* if

(2.1.2) $$|x_n - y_n| \leqq 2n^{-1} \quad (n \in \mathbb{Z}^+).$$

The set of real numbers is denoted by $\mathbb{R}$.

(2.2) **Proposition.** *Equality of real numbers is an equivalence relation.*

Proof: Parts (i) and (ii) of (1.1) are obvious. Part (iii) is a consequence of the following lemma.

(2.3) **Lemma.** *The real numbers $x \equiv (x_n)$ and $y \equiv (y_n)$ are equal if and only if for each positive integer j there exists a positive integer N_j such that*

(2.3.1) $$|x_n - y_n| \leqq j^{-1} \quad (n \geqq N_j).$$

Proof: If $x = y$, then (2.3.1) holds with $N_j \equiv 2j$.

Assume conversely that for each j in $\mathbb{Z}^+$ there exists N_j satisfying (2.3.1). Consider a positive integer n. If m and j are any positive integers with $m \geqq \max\{j, N_j\}$, then

$$\begin{aligned} |x_n - y_n| &\leqq |x_n - x_m| + |x_m - y_m| + |y_m - y_n| \\ &\leqq (n^{-1} + m^{-1}) + j^{-1} + (n^{-1} + m^{-1}) < 2n^{-1} + 3j^{-1}. \end{aligned}$$

Since this holds for all j in $\mathbb{Z}^+$, (2.1.2) is valid. □

Notice that the proof of Lemma (2.3) singles out a specific N_j satisfying (2.3.1). This situation is typical: every proof of a theorem which asserts the existence of an object must embody, at least implicitly, a finite routine for the construction of the object.

The rational number x_n is called the n^{th} *rational approximation* to the real number $x \equiv (x_n)$. Note that the operation from $\mathbb{R}$ to $\mathbb{Q}$ which takes the real number x into its n^{th} rational approximation is not a function.

For later use we wish to associate with each real number $x \equiv (x_n)$ an integer K_x such that

$$|x_n| < K_x \qquad (n \in \mathbb{Z}^+).$$

This is done by letting K_x be the least integer which is greater than $|x_1| + 2$. We call K_x the *canonical bound* for x.

The development of the arithmetic of the real numbers offers no surprises: we operate with real numbers by operating with their rational approximations.

(2.4) **Definition.** Let $x \equiv (x_n)$ and $y \equiv (y_n)$ be real numbers with respective canonical bounds K_x and K_y. Write

$$k \equiv \max\{K_x, K_y\}.$$

Let α be any rational number. We define

(a) $x + y \equiv (x_{2n} + y_{2n})_{n=1}^{\infty}$

(b) $xy \equiv (x_{2kn} y_{2kn})_{n=1}^{\infty}$

(c) $\max\{x, y\} \equiv (\max\{x_n, y_n\})_{n=1}^{\infty}$

(d) $-x \equiv (-x_n)_{n=1}^{\infty}$

(e) $\alpha^* \equiv (\alpha, \alpha, \alpha, \ldots)$.

(2.5) **Proposition.** *The sequences* $x + y$, xy, $\max\{x, y\}$, $-x$, *and* α^* *of Definition* (2.4) *are real numbers.*

Proof (a) Write $z_n \equiv x_{2n} + y_{2n}$ Then $x+y \equiv (z_n)$. For all positive integers m and n,

$$\begin{aligned}|z_m - z_n| &\leqq |x_{2m} - x_{2n}| + |y_{2m} - y_{2n}| \\ &\leqq (2n)^{-1} + (2m)^{-1} + (2n)^{-1} + (2m)^{-1} = n^{-1} + m^{-1}.\end{aligned}$$

Thus $x+y$ is a real number.

(b) Write $z_n \equiv x_{2kn} y_{2kn}$. Then $xy \equiv (z_n)$. For all positive integers m and n,

$$\begin{aligned}|z_m - z_n| &= |x_{2km}(y_{2km} - y_{2kn}) + y_{2kn}(x_{2km} - x_{2kn})| \\ &\leqq k|y_{2km} - y_{2kn}| + k|x_{2km} - x_{2kn}| \\ &\leqq k((2km)^{-1} + (2kn)^{-1} + (2km)^{-1} + (2kn)^{-1}) = n^{-1} + m^{-1}.\end{aligned}$$

Thus xy is a real number.

(c) Write $z_n \equiv \max\{x_n, y_n\}$ Then $\max\{x, y\} \equiv (z_n)$. Consider positive integers m and n. For simplicity assume that

$$x_m = \max\{x_m, x_n, y_m, y_n\}.$$

Then

$$\begin{aligned}|z_m - z_n| &= |x_m - \max\{x_n, y_n\}| \\ &= x_m - \max\{x_n, y_n\} \leqq x_m - x_n \leqq n^{-1} + m^{-1}.\end{aligned}$$

Thus $\max\{x, y\}$ is a real number.

(d) For all positive integers m and n,

$$|-x_m - (-x_n)| = |x_m - x_n| \leqq m^{-1} + n^{-1}.$$

Thus $-x$ is a real number

(e) This is obvious □

There is no trouble in proving that $(x, y) \mapsto x+y$, $(x, y) \mapsto xy$, and $(x, y) \mapsto \max\{x, y\}$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, that $x \mapsto -x$ is a function from $\mathbb{R}$ to $\mathbb{R}$; and that $\alpha \mapsto \alpha^*$ is a function from $\mathbb{Q}$ to $\mathbb{R}$.

The operation

$$x \mapsto |x| \equiv \max\{x, -x\}$$

is therefore a function from $\mathbb{R}$ to $\mathbb{R}$, and the operation

$$(x, y) \mapsto \min\{x, y\} \equiv -\max\{-x, -y\}$$

is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$

The next proposition states that the real numbers obey the same rules of arithmetic as the rational numbers.

(2.6) **Proposition.** *For arbitrary real numbers x, y, and z and rational numbers α and β,*

(a) $x+y = y+x$. $xy = yx$

(b) $(x+y)+z=x+(y+z), \quad x(yz)=(xy)z$

(c) $x(y+z)=xy+xz$

(d) $0^*+x=x, \quad 1^*x=x$

(e) $x-x=0^*$

(f) $|xy|=|x||y|$

(g) $(\alpha+\beta)^*=\alpha^*+\beta^*$, $(\alpha\beta)^*=\alpha^*\beta^*$, *and* $(-\alpha)^*=-\alpha^*$.

We omit the simple proofs of these results.

We shall use standard notations, such as $x+y+z$ and $\max\{x,y,z\}$, without further comment.

There are three basic relations defined on the set of real numbers. The first of these, the equality relation, has already been defined. The remaining relations, which pertain to order, are best introduced in terms of certain subsets $\mathbb{R}^+$ and $\mathbb{R}^{0+}$ of $\mathbb{R}$.

(2.7) **Definition.** A real number $x\equiv(x_n)$ is *positive*, or $x\in\mathbb{R}^+$, if

(2.7.1) $$x_n>n^{-1}$$

for some n in $\mathbb{Z}^+$. A real number $x\equiv(x_n)$ is *nonnegative*, or $x\in\mathbb{R}^{0+}$, if

(2.7.2) $$x_n\geqq -n^{-1} \qquad (n\in\mathbb{Z}^+).$$

The following criteria are often useful

(2.8) **Lemma.** *A real number $x\equiv(x_n)$ is positive if and only if there exists a positive integer N such that*

(2.8.1) $$x_m\geqq N^{-1} \qquad (m\geqq N).$$

A real number $x\equiv(x_n)$ is nonnegative if and only if for each n in $\mathbb{Z}^+$ there exists N_n in $\mathbb{Z}^+$ such that

(2.8.2) $$x_m\geqq -n^{-1} \qquad (m\geqq N_n).$$

Proof: Assume that $x\in\mathbb{R}^+$. Then $x_n>n^{-1}$ for some n in $\mathbb{Z}^+$. Choose N in $\mathbb{Z}^+$ with

$$2N^{-1}\leqq x_n-n^{-1}.$$

Then

$$x_m\geqq x_n-|x_m-x_n|\geqq x_n-m^{-1}-n^{-1}$$
$$\geqq x_n-n^{-1}-N^{-1}>N^{-1}$$

whenever $m\geqq N$. Therefore (2.8.1) is valid.

Conversely, if (2.8.1) is valid, then (2.7 1) holds with $n=N+1$ Therefore $x\in\mathbb{R}^+$.