Data Structures and Functional Programming	**Problem Set 5a**
CS 3110, Spring 2014	**Thursday, April 17**
Version: 2	**Last Modified: March 21, 2014**

Overview

In this assignment you will explore using logic to express and prove formal properties.

Objectives

This short assignment is designed to help you learn the following skills:

- Expressing intuitive properties using formal logic
- Understanding partial correctness specifications
- Developing proofs using natural deduction

Recommended Reading

- Lectures 14, 15, and 16
- Recitations 14 and 15

What to turn in

You should submit your solutions in a single file `logic.pdf`. Any comments you wish to make can go in `comments.txt` or `comments.pdf`. If you choose to submit any Karma work, you may submit the file `karma.pdf` (be sure to describe what you've done in the comments file).

Exercise 1:

Construct a proof tree for each of the following formulas:

(a) $P \wedge Q \Longrightarrow Q \wedge P$

(b) $(P \wedge Q \Longrightarrow R) \Longrightarrow (P \Longrightarrow (Q \Longrightarrow R))$

(c) $(P \vee Q \Longrightarrow R) \Longrightarrow (P \Longrightarrow R) \wedge (Q \Longrightarrow R)$

(d) $\exists x. \neg P(x) \Longrightarrow \neg(\forall x. P(x))$

Exercise 2:

Consider the following OCaml code:

```
(* pre: ???
 * post: nth [v0;...;vi;...;vn] i returns vi
*)
let rec nth (l:'a list) (i:int) =
  match i,l with
   | _,[] ->
     failwith "empty list"
   | i,_ when i < 0 ->
     failwith "negative index"
   | 0,h::_ ->
     h
   | _,_::t ->
     nth t (i - 1)
```

(a) Write a precondition that, when combined with the stated postcondition, yields a valid partial correctness specification for `nth`. For full credit, your precondition should impose as few constraints on the input as possible.

(b) Define a mapping f from `nth`'s inputs `(l,i)` to the natural numbers and briefly argue that this number decreases on each recursive call.

Exercise 3:

The following proofs are all incorrect. For each proof, give the names of valid instances of axioms and inference rules and write "Bogus" for invalid instances of axioms and inferences rules. For example, given the following (non)-proof,

$$\dfrac{\dfrac{\dfrac{}{P \vdash P}\,(1) \quad \dfrac{}{P \vdash Q}\,(2)}{P \vdash P \wedge Q}\,(3)}{\vdash P \Longrightarrow P \wedge Q}\,(4)$$

you would write:

(1) Assumption

(2) Bogus

(3) $\wedge$-introduction

(4) $\Longrightarrow$-introduction

(a)
$$\dfrac{\dfrac{\dfrac{}{P \vee Q \vdash P}\,(1) \quad \dfrac{}{P \vee Q \vdash Q}\,(2)}{P \vee Q \vdash P \wedge Q}\,(3)}{\vdash P \vee Q \Longrightarrow P \wedge Q}\,(4)$$

(b)
$$\dfrac{\dfrac{}{P \vdash P}\,(1) \quad \dfrac{\dfrac{\dfrac{}{P \vdash P}\,(2) \quad \dfrac{}{P \vdash P \Longrightarrow \bot}\,(3)}{P \vdash \bot}\,(4)}{P \vdash \neg P}\,(5)}{\vdash P \Longrightarrow P \wedge \neg P}\,(6)$$

(c)
$$\dfrac{\dfrac{}{\exists x.P(x) \vdash \exists x.P(x)}\,(1) \quad \dfrac{\dfrac{\dfrac{}{\exists x.P(x), P(a) \vdash P(a)}\,(2)}{\exists x.P(x), P(a) \vdash \forall x.P(x)}\,(3)}{\exists x.P(x) \vdash P(a) \Longrightarrow \forall x.P(x)}\,(4)}{\dfrac{\exists x.P(x) \vdash \forall x.P(x)}{\vdash \exists x.P(x) \Longrightarrow \forall x.P(x)}\,(6)}\,(5)$$

Karma suggestions

- Prove $(P \implies (Q \implies R)) \implies (P \wedge Q \implies R)$
- Prove $(P \implies R) \wedge (Q \implies R) \implies (P \vee Q \implies R)$
- Prove that your precondition for nth yields a valid partial correctness specification.
- Prove that your precondition for nth yields a valid total correctness specification.

Comments

We would like to know how this assignment went for you. Were there any parts that you didn't finish or wish you had done in a better way? Which parts were particularly fun or interesting? Did you do any Karma problems?