Induction (continued)

Meir Friedenberg

- More ways things can go wrong with induction
- Inductive Definitions

Theorem: Every integer $n > 1$ is a product of prime numbers.

Theorem: Every integer $n > 1$ is a product of prime numbers.

Proof: By strong induction. (On the board.)

But now what about…

Theorem: Every integer $n > 1$ can be written as a product of prime numbers in a unique way.

This is a true theorem. But let's try to prove it by strong induction. (Warning: things will go wrong…)

But now what about…

Theorem: Every integer $n > 1$ can be written as a product of prime numbers in a unique way.

This is a true theorem. But let's try to prove it by strong induction. (Warning: things will go wrong…)

$\vdots$

$$36 = 3 \times 12 = 3p_1p_2p_3$$
$$36 = 4 \times 9 = q_1q_2r_1r_2$$

How do you know that $3p_1p_2p_3 = q_1q_2r_1r_2$??

This is a *breakdown error.*

If you're trying to show something is unique, and in order to do so you break down an object, you need to show that nothing would change if you broke it down differently.

The actual proof of this theorem is much more subtle.

Definition: A regular n-gon is a polygon with n sides of all equal length and n angles of all equal angle.

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$**.**

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

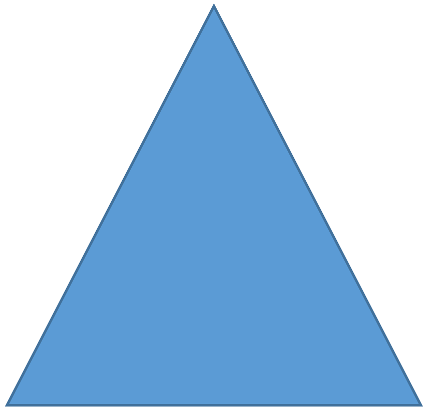
Proof: By induction.

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

Proof: By induction.

Base Case:

You learned this in high school.

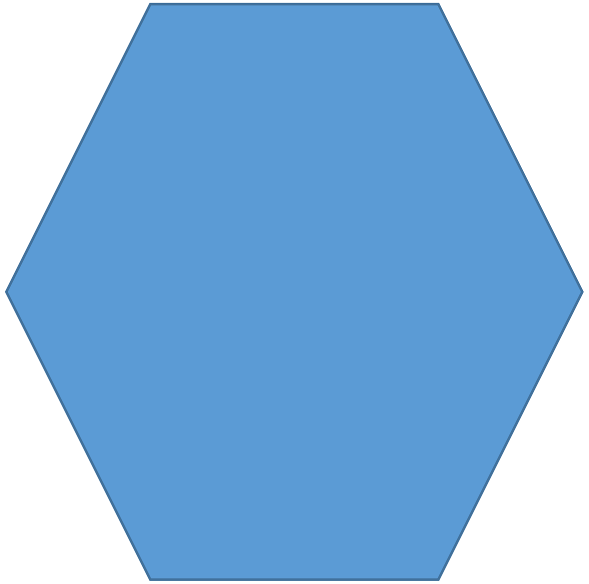


Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

Proof: By induction.

Inductive Step: Assume P(n).

Consider a regular (n+1)-gon.



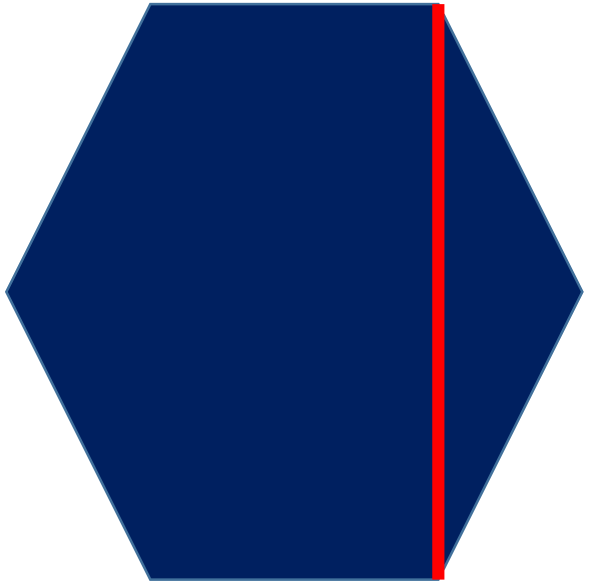
Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

Proof: By induction.

Inductive Step: Assume P(n).

Consider a regular (n+1)-gon.

Lop off one of the corners



Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$**.**

Proof: By induction.

Inductive Step: Assume P(n).

Consider a regular (n+1)-gon.

Lop off one of the corners.

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

Proof: By induction.

Inductive Step: Assume P(n).

Consider a regular (n+1)-gon.

Lop off one of the corners.

By the IH, the regular n-gon has $180(n-2)$ degrees. The triangle has 180.

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$**.**

Proof: By induction.

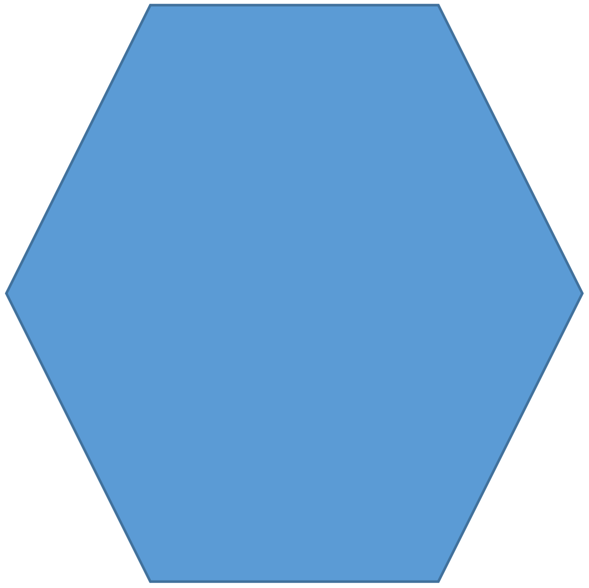
Inductive Step: Assume P(n).

Consider a regular (n+1)-gon.

Lop off one of the corners.

By the IH, the regular n-gon has

$180(n-2)$ degrees. The triangle has 180. Glue them back together, get $180(n-1)$, as desired.



Where did we cheat?

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

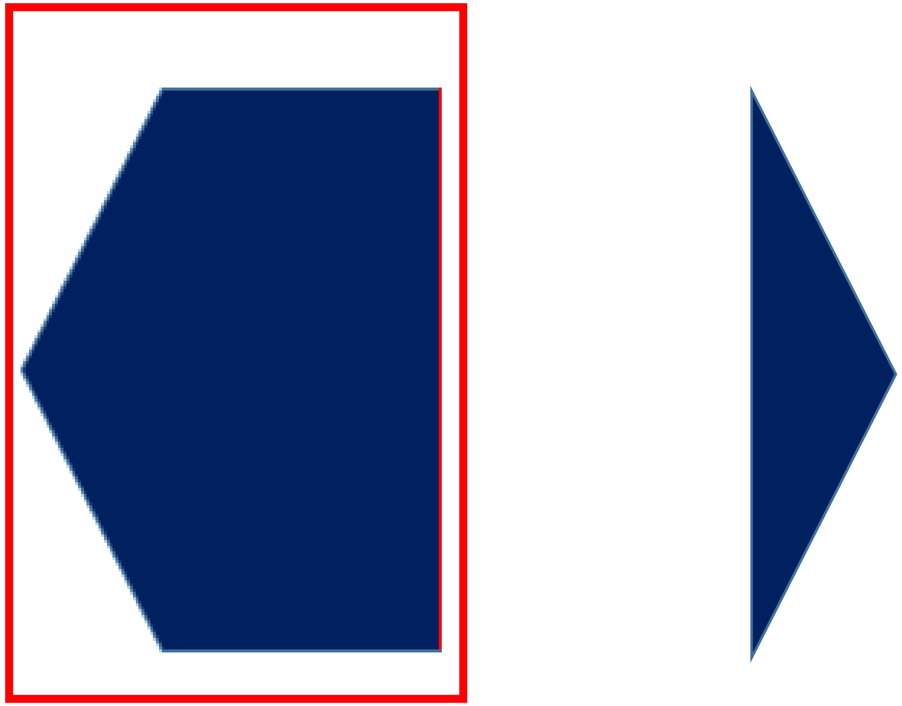
Proof: By induction.

Inductive Step: Assume P(n).

Consider a regular (n+1)-gon.

Lop off one of the corners.

By the IH, the regular n-gon has $180(n-2)$ degrees. The triangle has 180.



How do we fix this?

How do we fix this?

<u>Strengthen the induction hypothesis.</u>

Let $p(n)$ be "the sum of the internal angles of *any* n-gon is $180(n-2)$."

- More ways things can go wrong with induction
- Inductive Definitions

Inductive Definitions

Kind of like how you can define an object recursively while programming, in math you can define things inductively!

Inductive Definitions

$$\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$$

is a little ambiguous because of the ...

Inductive Definitions

$$\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$$

is a little ambiguous because of the ...

To be fully formal, define it inductively:

$$\sum_{k=1}^{1} a_k = a_1$$
$$\sum_{1}^{n+1} a_k = a_{n+1} + \sum_{k=1}^{n} a_k$$

Inductive Definitions: Factorials!

$$0! = 1$$
$$(n+1)! = (n+1) \times n!$$

Inductive Definitions: Propositional Formulae

Let ϕ_0 be the set of primitive propositions.

Let $\phi_{n+1} = \phi_n \cup \{ \neg\varphi, \varphi \wedge \psi : \varphi, \psi \in \phi_n\}$

The set ϕ^* of propositional formulae is $\bigcup_{n=0}^{\infty} \phi_n$.

Inductive Definitions: Transitive Closure

Given a relation R on a set S, we can define the transitive closure of R inductively:

- Let $R_0 = R$
- Let $R_{n+1} = R_n \cup \{(s,t) : \exists u \in S \text{ s.t. } ((s,u) \in R_n \text{ and } (u,t) \in R_n)\}$
- Let $R' = \bigcup_{n=0}^{\infty} R_n$

Theorem: R' is the transitive closure of R.

Inductive Definitions: Transitive Closure

Theorem: R' is the transitive closure of R.

How would you prove this? Show

- $R \subseteq R'$
- R' is transitive
- For any R'', if $R \subseteq R''$ and R'' is transitive then $R' \subseteq R''$ (i.e. R' is the smallest transitive set containing R)

Showing this will apparently be a homework problem.

Inductive Definitions: Fibonacci Numbers

- [Leonardo of Pisa, 12^{th} century] Was interested in the growth of rabbit populations. As a mathematician, also liked elegant (if oversimplified) models.
- Suppose you start with a breeding pair of rabbits. After two months, they produce a new breeding pair of rabbits as offspring. Rabbits never die, and every breeding pair produces a new breeding pair every month after the first. How many rabbits do you have after n months?

Inductive Definitions: Fibonacci Numbers

Let f_n be the number of pair after n months.

By assumption $f_1 = f_2 = 1$.

For $n > 2$, $f_{n+1} = f_n + f_{n-1}$

pairs last month

new pairs = num pairs two months ago

1, 1, 2, 3, 5, 8…

Fibonacci numbers have lots of nice properties. Here's one:

Let $r = \frac{1+\sqrt{5}}{2}$

Claim: $f_n \geq r^{n-2}$ for all n.

Fibonacci numbers have lots of nice properties. Here's one:

Let $r = \frac{1+\sqrt{5}}{2}$ ← The Golden Ratio. Positive solution to $r^2 = r + 1$.

Claim: $f_n \geq r^{n-2}$ for all n.

Fibonacci numbers have lots of nice properties. Here's one:

Let $r = \frac{1+\sqrt{5}}{2}$ ← The Golden Ratio. Positive solution to $r^2 = r + 1$.

Claim: $f_n \geq r^{n-2}$ for all n.

Proof: By induction. (On the board.)

Appendix

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \dots p_j = q_1 \dots q_k$.

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \dots p_j = q_1 \dots q_k$.

p_1 divides n, so there must be some i such that p_1 divides q_i.

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \dots p_j = q_1 \dots q_k$.

p_1 divides n, so there must be some i such that p_1 divides q_i.

But since p_1 and q_i are both prime, this implies that $p_1 = q_i$.

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \ldots p_j = q_1 \ldots q_k$.

p_1 divides n, so there must be some i such that p_1 divides q_i.

But since p_1 and q_i are both prime, this implies that $p_1 = q_i$.

But then we could divide $p_1 \ldots p_j$ by p_1 and $q_1 \ldots q_k$ by q_i to get a new number, less than n, with two prime factorizations. This gives us a contradiction.

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \dots p_j = q_1 \dots q_k$.

p_1 divides n, so there must be some i such that p_1 divides q_i.

But since p_1 and q_i are both prime, this implies that $p_1 = q_i$.

But then we could divide $p_1 \dots p_j$ by p_1 and $q_1 \dots q_k$ by q_i to get a new number, less than n, with two prime factorizations. This gives us a contradiction.

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \dots p_j = q_1 \dots q_k$.

p_1 divides n, so there must be some i such that p_1 divides q_i.

Euclid's Lemma

But since p_1 and q_i are both prime, this implies that $p_1 = q_i$.

But then we could divide $p_1 \dots p_j$ by p_1 and $q_1 \dots q_k$ by q_i to get a new number, less than n, with two prime factorizations. This gives us a contradiction.

The Fundamental Theorem of Arithmetic

Proof: Let n be the minimal number such that it has two distinct prime factorizations, so $n = p_1 \ldots p_j = q_1 \ldots q_k$.

p_1 divides n, so there must be some i such that p_1 divides q_i.

Euclid's Lemma

But since p_1 and q_i are both prime, this implies that $p_1 = q_i$.

But then we could divide $p_1 \ldots p_j$ by p_1 and $q_1 \ldots q_k$ by q_i to get a new number, less than n, with two prime factorizations. This gives us a contradiction.

I'm not going to prove Euclid's Lemma today, but feel free to look it up – the standard proof uses another important lemma called Bézout's identity.