CS 2802: Homework 3

February 3, 2020

Handed out Feb. 3; due Feb. 10

- Read Chapter 5
- Do the following problems:
  - Problem 1: Given a relation R on a set S, recall that for $s \in S$, we can define $[s]_R = \{s' : (s, s') \in R$. That is, $[s]_R$ consists of all the elements in S related to s by R¿ Show that R is an equivalence relation iff (a) the sets $[s]_R$ form a partition of S (i.e., for all $s, s' \in S$, we have either $[s] = [s']$ or $[s] \cap [s'] = \emptyset$) and (b) $s \in [s]_R$ for all $s \in S$. (In the last homework, you showed that if R is an equivalence relation, then the sets $[s]_R$ form a partition. You don't have to reprove that. So for this week, you just have to check that if R is an equivalence relation, then $s \in [s]_R$, and that if R has properties (a) and (b) above, then R is an equivalence relation.)
  - Problem 2: Show that if f is a bijection from A to B and g is a bijection from B to C, then $g \circ f$ is a bijection from A to C.
  - Problem 3: Show that if $S \neq \emptyset$, then $f : S \rightarrow T$ is an injection iff f has a left inverse.
  - Problem 4: Show that if A and B are countable, then so is $A \cup B$. [You may also want to think about the more general question: How does the cardinality of $A \cup B$ compared to the cardinality of A and B? You don't have to hand this in though.]
  - Problem 5: If $A_0, A_1, A_2, \ldots$ are all countably infinite and disjoint (i.e., $A_i \cap A_j = \emptyset$), construct a bijection between $\cup_{i=0}^{\infty} A_i$ and $I\!N \times I\!N$ ($I\!N$ is the natural numbers). Carefully prove that your construction works.
  - Problem 6: (Based on Problem 8.16 in MCS.) In this problem you'll prove what might seem quite surprising: there is a a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$. (Some notation: (a, b) denotes the interval $\{x \in I\!R : a < x < b\}$; similarly, $(a, b] = \{x \in I\!R : a < x \leq b\}$; $[a, b) = \{x \in I\!R : a \leq x < b\}$; $(a, \infty) = \{x \in I\!R : a < x\}$; and $[a, \infty) = \{x \in I\!R : a \leq x\}$.)

(a) Describe a bijection from $(0,1]$ to $[0,\infty)$. (Hint: $1/x$ almost works.)

b) An infinite sequence of the decimal digits $\{0,1,\ldots,9\}$ will be called *long* if it does not end with all 0s. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all long sequences. Describe a bijection from L to the half-open real interval $(0,1]$. (Hint: Put a decimal point at the beginning of the sequence.)

(c) Describe an injection from L to L^2 and an injection from L^2 to L. (For the second injection, consider alternating digits.)

(d) Show that if there is a bijection from A to B, then there is a bijection from $A \times A$ to $B \times B$.

(e) Complete the argument (i.e., show that there is a bijection from $(0,1]$ to $[0,\infty)\times[0,\infty)$) using the Schröder-Bernstein Theorem and Extra Problem 4 from HW2.

Note that the parts of this problem are independent. For example, you can do (d) even if you don't do (a), (b), and (c). When you're doing a later part, you can use the earlier parts, even if you didn't prove them.

– Problem 7: Give a bijection between infinite binary sequences (i.e., infinite sequences of 0s and 1s) and subsets of the natural numbers.

- Challenge problem #1 (you don't have to hand in challenge problems): Prove that there is a bijection between the real numbers in $[0,1]$ and subsets of the natural numbers. (Hint: problem #7 is useful here.)

- Challenge problem #2:

  (a) (An easy warmup:) Find an explicit bijection $f : \mathbb{N} \cup \{a,b\} \to \mathbb{N}$

  (b) Find an explicition bijection $f : [0,1] \to (0,1)$. (It easily follows from the Schröder-Bernstein Theorem that there is a bijection, but constructing it requires a bit of thought. Hint: use the ideas of part (a).)