Patterns and Finite Automata

A *pattern* is a set of objects with a recognizable property.

- In computer science, we're typically interested in patterns that are sequences of character strings
  - I think "Halpern" a very interesting pattern
  - I may want to find all occurrences of that pattern in a paper
- Other patterns:
  - **if** followed by any string of characters followed by **then**
  - all filenames ending with ".doc"

Pattern matching comes up all the time in text search.

A *finite automaton* is a particularly simple computing device that can recognize certain types of patterns, called *regular languages*

- The text does not cover finite automata; there is a separate handout on CMS.

Finite Automata

A *finite automaton* is a machine that is always in one of a finite number of states.

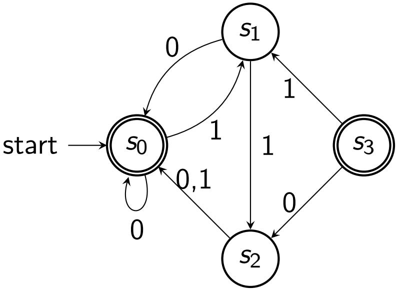
- When it gets some input, it moves from one state to another
  - If I'm in a "sad" state and someone hugs me, I move to a "happy" state
  - If I'm in a "happy" state and someone yells at me, I move to a "sad" state
- **Example:** A digital watch with "buttons" on the side for changing the time and date, or switching it to "stopwatch" mode, is an automaton
  - What are the states and inputs of this automaton?
- A certain state is denoted the *start* state
  - That's how the automaton starts life
- Other states are denoted *final* state
  - The automaton stops when it reaches a final state
  - (A digital watch has no final state, unless we count running out of battery power.)

Representing Finite Automata Graphically

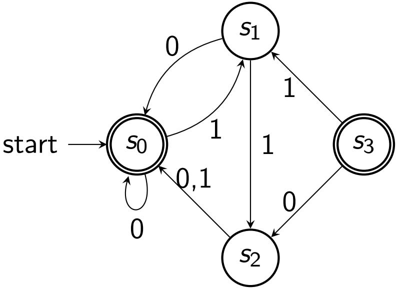
A finite automaton can be represented by a labeled directed graph.

- The nodes represent the states of the machine
- The edges are labeled by inputs, and describe how the machine transitions from one state to another

Example:



- There are four states: s_0, s_1, s_2, s_3
  - s_0 is the start state (denote by "start $\rightarrow$", by convention)
  - s_0 and s_3 are the final states (denoted by double circles, by convention)
- The labeled edges describe the transitions for each input
  - The inputs are either 0 or 1
    - in state s_0 and reads 0, it stays in s_0
    - If the machine is in state s_0 and reads 1, it moves to s_1
    - If the machine is in state s_1 and reads 0, it moves to s_1
    - If the machine is in state s_1 and reads 1, it moves to s_2



What happens on input 00000? 0101010? 010101? 11?

- Some strings move the automaton to a final state; some don't.
- The strings that take it to a final state are *accepted*.

A Parity-Checking Automaton

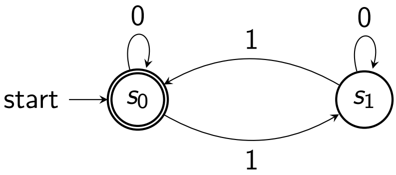
Here's an automaton that accepts strings of 0s and 1s that have even parity (an even number of 1s).

We need two states:

- s_0: we've seen an even number of 1s so far
- s_1: we've seen an odd number of 1s so far

The transition function is easy:

- If you see a 0, stay where you are; the number of 1s hasn't changed
- If you see a 1, move from s_0 to s_1, and from s_1 to s_0

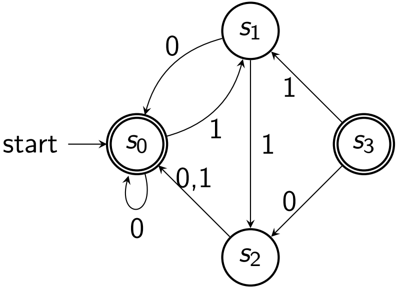


Finite Automata: Formal Definition

A *(deterministic) finite automaton* is a tuple $M = (S, I, f, s_0, F)$:

- S is a finite set of states;
- I is a finite input alphabet (e.g. $\{0, 1\}$, $\{a, \ldots, z\}$)
- f is a transition function; $f : S \times I \to S$
  - f describes what the next state is if the machine is in state s and sees input $i \in I$.
- $s_0 \in S$ is the initial state;
- $F \subseteq S$ is the set of final states.

Example:



- $S = \{s_0, s_1, s_2, s_3\}$
- $I = \{0, 1\}$
- $F = \{s_0, s_3\}$
- The transition function f is described by the graph;
  - $f(s_0, 0) = s_0$; $f(s_0, 1) = s_1$; $f(s_1, 0) = s_0$; ...

You should be able to translate back and forth between finite automata and the graphs that describe them.

Describing Languages

The *language* accepted (or *recognized*) by an automaton is the set of strings that it accepts.

- A *language* is a set of strings

We need tools for describing languages.

- If A and B are sets of strings, then AB, the *concatenation* of A and B, is $\{ab : \ a \in A,\ b \in B\}$.
  - **Example:** If $A = \{0, 11\}$, $B = \{111, 00\}$, then
    - $AB = \{0111, 000, 11111, 1100\}$
    - $BA = \{1110, 11111, 000, 0011\}$
- Define A^{n+1} inductively:
  - $A^0 = \{\lambda\}$: λ is the empty string
    - $\lambda x = x\lambda = x$ for all strings x
  - $A^1 = A$
  - $A^{n+1} = AA^n$
- $A^* = \cup_{n=0}^{\infty} A^n$.

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  - $A^1 = A$
  - $A^{n+1} = AA^n$
- $A^* = \cup_{n=0}^{\infty} A^n$.
  - What's $\{0, 1\}^n$? $\{0, 1\}^*$? $\{11\}^*$?

Regular Expressions

A *regular expression* is an algebraic way of defining a pattern

Definition: The set of *regular expressions over I* (where I is an input set) is the smallest set S of expressions such that:

- the symbol $\emptyset \in S$ (that should be a boldface $\emptyset$)
- the symbol $\lambda \in S$ (that should be a boldface λ)
- the symbol $\mathbf{x} \in S$ is a regular expression if $x \in I$;
- if $\mathbf{E}_1$ and $\mathbf{E}_2$ are in S, then so are $(\mathbf{E_1 E_2})$, $(\mathbf{E_1} \cup \mathbf{E_2})$ and $\mathbf{E_1^*}$.

That is, we start with the empty set, λ, and elements of I, then close off under union, concatenation, and $*$.

- A regular set is a *syntactic* object: a sequence of symbols.
- Concatenation, union, and $*$ are overloaded; they're used for both languages (sets of strings) and regular expressions (sequences of symbols)
- The parens help disambiguate: $((\mathbf{ab}) \cup \mathbf{c}) \neq (\mathbf{a}(\mathbf{b} \cup \mathbf{c}))$
- There is an equivalent inductive definition (see homework).

Those of you familiar with the programming language Perl or Unix searches should recognize the syntax ...

Each regular expression $\mathbf{E}$ over I defines a subset of I^*, denoted $L(\mathbf{E})$ (the *language* of $\mathbf{E}$) in the obvious way:

- $L(\emptyset) = \emptyset$;
- $L(\lambda) = \{\lambda\}$;
- $L(\mathbf{x}) = \{x\}$;
- $L(\mathbf{E_1 E_2}) = L(\mathbf{E_1})L(\mathbf{E_2})$;
- $L(\mathbf{E_1} \cup \mathbf{E_2}) = L(\mathbf{E_1}) \cup L(\mathbf{E_2})$;
- $L(\mathbf{E}^*) = L(E)^*$.

Examples:

- What's $L(\mathbf{0^*10^*10^*})$?
- What's $L((\mathbf{0^*10^*10^*})^\mathbf{n})$? $L(\mathbf{0}^*(\mathbf{0^*10^*10^*})^*)$?
- $L(\mathbf{0}^*(\mathbf{0^*10^*10^*})^*)$ is the language accepted by the parity automaton!
- If $\Sigma = \{a, \ldots, z, A, \ldots, Z, 0, \ldots, 9\} \cup \mathit{Punctuation}$, what is $\Sigma^*\{H\}\{a\}\{l\}\{p\}\{e\}\{r\}\{n\}\Sigma^*$?
  - *Punctuation* consists of the punctuation symbols (comma, period, space, etc.)
  - Σ is the input alphabet
  - Note that $L(\mathbf{\Sigma^* Halpern \Sigma^*}) = \Sigma^*\{H\}\{a\}\{l\}\{p\}\{e\}\{r\}\{n\}\Sigma^*$.

Can you define an automaton that accepts exactly the strings in $\Sigma^* Halpern \Sigma^*$?

- How many states would you need?

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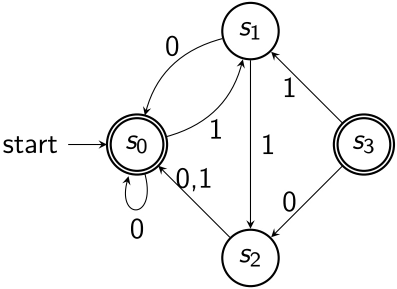
What language is represented by the automaton in the original example:



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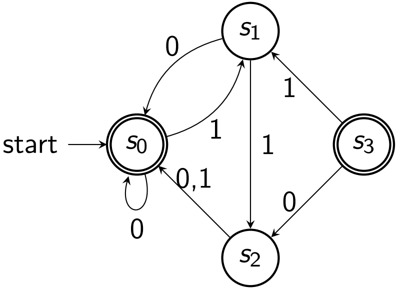
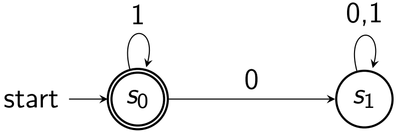


- $((10)^*0^*((110) \cup (111))^*)^*$
- Perhaps clearer: $((0 \cup 1)^*0 \cup 111)^*$
- It's not easy to prove this formally!

What language is accepted by the following automata:



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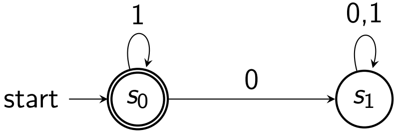
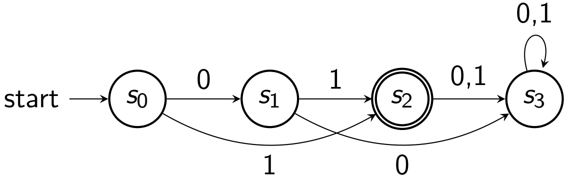


$L(\mathbf{1}^*) = \{1\}^*$



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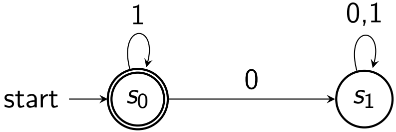

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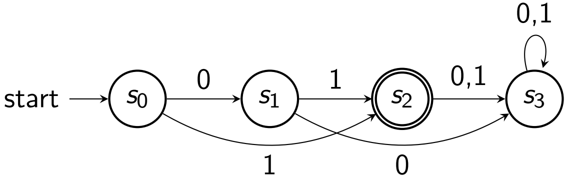
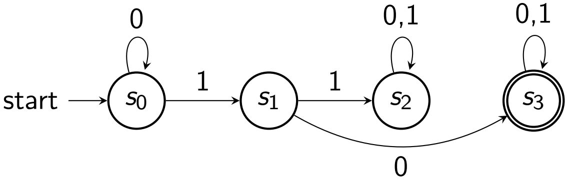
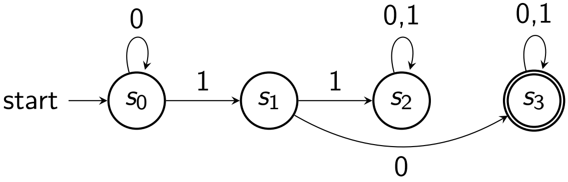


$L(\mathbf{1} \cup \mathbf{01}) = \{1, 01\}$


0
0,1
0,1
1
1
start
s_0
s_1
s_2
s_3
0



$L(\mathbf{0}^*\mathbf{10}(\mathbf{0} \cup \mathbf{1})^*) = \{0\}^*\{10\}\{0,1\}^*$

Nondeterministic Finite Automata

So far we've considered *deterministic* finite automata (DFA)

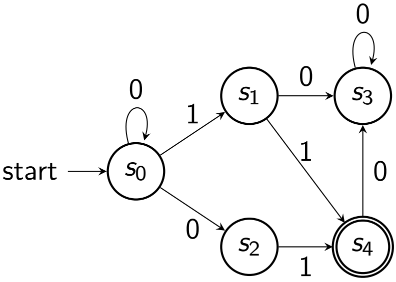
- what happens in a state is completely determined by the input symbol read

Nondeterministic finite automata allow several possible next states when an input is read.

Formally, a nondeterministic finite automaton is a tuple $M = (S, I, f, s_0, F)$. All the components are just like a DFA, except now $f : S \times I \to 2^S$ (before, $f : S \times I \to S$).

- if $s' \in f(s, i)$, then s' is a possible next state if the machines is in state s and sees input i.

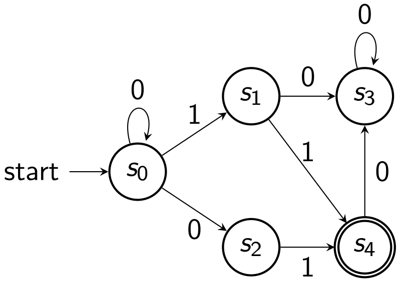
We can still use a graph to represent an NFA. There might be several edges coming out of a state labeled by $i \in I$, or none. In the example below, there are two edges coming out of s_0 labeled 0, and none coming out of s_4 labeled 1.



- Can either stay in s_0 or move to s_2
- On input 111, get stuck in s_4 after 11, so 111 not accepted.

- An NFA M *accepts* (or *recognizes*) a string x if it is possible to get to a final state from the start state with input x.
- The language L is accepted by an NFA M consists of all strings accepted by M.

What language is accepted by this NFA:



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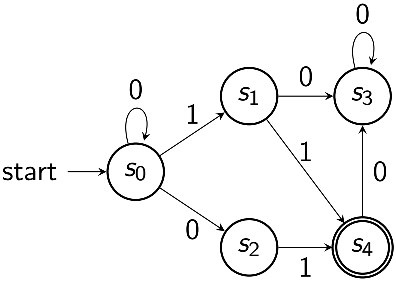


$L(\mathbf{0}^*\mathbf{01} \cup \mathbf{0}^*\mathbf{11})$

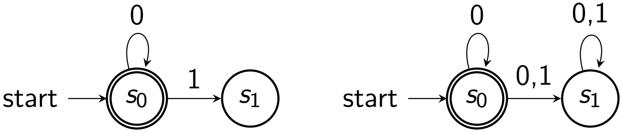
Equivalence of Automata

Every DFA is an NFA, but not every NFA is a DFA.

- Do we gain extra power from nondeterminism?
  - Are there languages that are accepted by an NFA that can't be accepted by a DFA?
  - Somewhat surprising answer: NO!

Define two automata to be *equivalent* if they accept the same language.

Example:



Theorem: Every nondeterministic finite automaton is equivalent to some deterministic finite automaton.

Proof: Given an NFA $M = (S, I, f, s_0, F)$, let $M' = (S', I, f', \{s_0\}, F')$, where

- $S' = 2^S$
- $f'(A, i) = \{t : t \in f(s, i) \text{ for some } s \in A\} \in 2^S$
  - $f' : 2^S \times I \to 2^S$ (i.e., $f' : S' \times I \to S'$)
- $F' = \{A : A \cap F \neq \emptyset\}$

Thus,

- the states in M' are subsets of states in M;
- the final states in M' are the sets which contain a final state in M;
- in state A, given input i, the next state consists of all possible next states from an element in A.

M' is *deterministic*.

- This is called the *subset* construction.
- The states in M' are subsets of states in M.

We want to show that M accepts x iff M' accepts x.

- Let $x = x_1 \dots x_k$.
- If M accepts x, then there is a sequence of states $s_0, \dots, s_k$ such that $s_k \in F$ and $s_{i+1} \in f(s_i, x_{i+1})$.
  - That's what it means for an NFA M to accept x
  - $s_0, \dots, s_k$ is a possible sequence of states that M goes through on input x
    - It's only one possible sequence: M is an NFA
- Define $A_0, \dots, A_k$ inductively:
  $A_0 = \{s_0\}$ and $A_{i+1} = f'(A_i, x_{i+1})$.
  - Intuitively, A_i is the set of states that M could be in after seeing $x_1 \dots x_i$
    - Remember: a state in M' is a set of states in M.
    - M' is deterministic: this sequence is unique.
  - An easy induction shows that $s_i \in A_i$.
  - Therefore $s_k \in A_k$, so $A_k \cap F \neq \emptyset$.
  - Conclusion: $A_k \in F'$, so M' accepts x.

For the converse, suppose that M' accepts x

- Let $A_0, \ldots, A_k$ be the sequence of states that M' goes through on input x.
- Since $A_k \cap F \neq \emptyset$, there is some $t_k \in A_k \cap F$.
- By induction, if $1 \leq j \leq k$, can find $t_{k-j} \in A_{k-j}$ such that $t_{k-j+1} \in f(t_{k-j}, x_{k-j})$.
- Since $A_0 = \{s_0\}$, we must have $s_0 = t_0$.
- Thus, $t_0 \ldots t_k$ is an *accepting path* for x in M
- Conclusion: M accepts x

Notes:

- Michael Rabin and Dana Scott won a Turing award for defining NFAs and showing they are equivalent to DFAs
- This construction blows up the number of states:
  - $|S'| = 2^{|S|}$
  - Sometimes you can do better; in general, you can't

Regular Languages and Finite Automata

Some notation:

- Language L is *regular* iff $L = L(\mathbf{E})$ for some regexp $\mathbf{E}$.
- $L(M)$ is the language accepted by the automaton M

Theorem: $L = L(M)$ for some automaton M iff L is regular.

First we'll show that every regular language is accepted by some finite automaton:

Proof: We show that $L(\mathbf{E})$ is accepted by a finite automaton by induction on the (length/structure) of $\mathbf{E}$. We need to show that

- $\emptyset = L(\emptyset) = L(M)$ for some finite automaton M
  - Easy: build an automaton where no input ever reaches a final state
- $\{\lambda\} = L(\lambda) = L(M)$ for some finite automaton M
  - Easy: M has two states, s_0 and s_1, s_0 is the only accepting state, but every non-empty string ends leads to s_1.
- For each $x \in I$, $\{x\} = L(\mathbf{x}) = L(M)$ for some automaton M
  - Easy: an automaton with states $\{s_0, s_1, s_2\}$, only s_1 is an accepting state, x leads from s_0 to s_1, all other nonempty strings lead to s_2.

We next show that $L(\mathbf{E_1 E_2})$ is accepted by some automaton.

Suppose that $L(\mathbf{E_1}) = A$, $L(\mathbf{E_2}) = B$. By the induction hypothesis, there exist automata $M_A = (S_A, I, f_A, s_A, F_A)$ and $M_B = (S_B, I, f_B, s_B, F_B)$ such that $A = L(M_A)$ and $B = L(M_B)$.

Suppose that M_A and M_B and NFAs, and S_A and S_B are disjoint (without loss of generality).

Idea: We hook M_A and M_B together. Let NFA

$M_{AB} = (S_A \cup S_B, I, f_{AB}, s_A, F_{AB})$, where

- $F_{AB} = \begin{cases} F_B \cup F_A & \text{if } \lambda \in B; \\ F_B & \text{otherwise} \end{cases}$
- $t \in f_{AB}(s, i)$ if either
  - $s \in S_A$ and $t \in f_A(s, i)$, or
  - $s \in S_B$ and $t \in f_B(s, i)$, or
  - $s \in F_A$ and $t \in f_B(s_B, i)$ ("switch" from M_A to M_B)

Idea: given input $xy \in AB$, the machine "guesses" when to switch from running M_A to running M_B.

Claim: $L(M_{AB}) = AB$.

Proof: There are two parts to this proof:

1. Showing that if $x \in AB$, then $x \in L(M_{AB})$.
2. Show that if $x \in L(M_{AB})$, then $x \in AB$.

For part 1, suppose that $x = ab \in AB$, where $a = a_1 \dots a_k$ and $b = b_1 \dots b_m$. Then there exists an *accepting path* for a in M_A and one for b in M_B; that is, a sequence of states $s_0, \dots, s_k \in S_A$ and a sequence of states $t_0, \dots, t_m \in S_B$ such that

- $s_0 = s_A$ and $t_0 = s_B$;
- $s_{i+1} \in f_A(s_i, a_{i+1})$ and $t_{i+1} \in f_B(t_i, b_{i+1})$
- $s_k \in F_A$ and $t_m \in F_B$.

That means that after reading a, M_{AB} could be in state s_k. If $b = \lambda$, M_{AB} accepts a (since $s_k \in F_A \subseteq F_{AB}$ if $\lambda \in B$). Otherwise, M_{AB} can continue to $t_1, \dots, t_m$ when reading b, so it accepts ab (since $t_m \in F_B \subseteq F_{AB}$).

- is, $s_0, \dots, s_k, t_1, \dots, t_m$ is an accepting path for ab
  - Note that there is no t_0; we go from s_k to t_1

For part 2, suppose that $x = c_1 \ldots c_n$ is accepted by M_{AB}. That means that there is a sequence of states $s_0, \ldots, s_n \in S_A \cup S_B$ such that

- $s_0 = s_A$
- $s_{i+1} \in f_{AB}(s_i, c_{i+1})$
- $s_n \in F_{AB}$

If $s_n \in F_A$, then $\lambda \in B$, $s_0, \ldots, s_n \subseteq S_A$ (since once M_{AB} moves to a state in S_B, it never moves to a state in S_A), so x is accepted by M_A. Thus, $x \in A \subseteq AB$.

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If $s_n \in F_B$, let s_j be the first state in the sequence in S_B. Then $s_0, \ldots, s_{j-1} \subseteq S_A$, $s_{j-1} \in F_A$, so $c_1 \ldots c_{j-1}$ is accepted by M_A, and hence is in A. Moreover, $s_B, s_j, \ldots, s_n \subseteq S_B$ (once M_{AB} is in a state of S_B, it never moves to a state of S_A), so $c_j \ldots c_n$ is accepted by M_B, and hence is in B. Thus, $x = (c_1 \ldots c_{j-1})(c_j \ldots c_n) \in AB$.

We next show that $L(\mathbf{E_1} \cup \mathbf{E_2})$ is accepted by some automaton.

- Suppose that $A = L(\mathbf{E}_1)$ and $B = \mathbf{E}_2)$.
- By the induction hypothesis, there exist automata $M_A = (S_A, I, f_A, s_A, F_A)$ and $M_B = (S_B, I, f_B, s_B, F_B)$ such that $A = L(M_A)$ and $B = L(M_B)$.
- Again, assume without loss of generality that M_A and M_B and NFAs, and that S_A and S_B are disjoint.

Idea: given input $x \in A \cup B$, the machine "guesses" whether to run M_A or M_B.

- $M_{A \cup B} = (S_A \cup S_B \cup \{s_0\}, I, f_{A \cup B}, s_0, F_{A \cup B})$, where
  - s_0 is a new state, not in $S_A \cup S_B$
  - $f_{A \cup B}(s, i) = \begin{cases} f_A(s, i) & \text{if } s \in S_A \\ f_B(s, i) & \text{if } s \in S_B \\ f_A(s_A, i) \cup f_B(s_B, i) & \text{if } s = s_0 \end{cases}$
  - $F_{A \cup B} = \begin{cases} F_A \cup F_B \cup \{s_0\} & \text{if } \lambda \in A \cup B \\ F_A \cup F_B & \text{otherwise.} \end{cases}$
- We have to prove that $L(M_{A \cup B}) = A \cup B$; this is straightforward.

Last step: show that $L(\mathbf{E}^*)$ is regular.

As before, suppose that $A = L(\mathbf{E})$, and that $M_A = S_A, I, f_A, s_A, F_A)$ accepts M.

$M_{A^*} = (S_A \cup \{s_0\}, I, f_{A^*}, s_0, F_A \cup \{s_0\})$, where

- s_0 is a new state, not in S_A;
- $f_{A^*}(s,i) = \begin{cases} f_A(s,i) & \text{if } s \in S_A - F_A; \\ f_A(s,i) \cup f_A(s_A,i) & \text{if } s \in F_A; \\ f_A(s_A,i) & \text{if } s = s_0 \end{cases}$

We now have to prove that $L(M_{A^*}) = A^*$.

- Left for homework!

Next we'll show that every language accepted by a finite automaton is regular:

Proof: Fix an automaton M with states $\{s_0, \ldots, s_n\}$. Can assume wlog (without loss of generality) that M is deterministic.

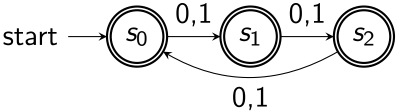
- a language is accepted by a DFA iff it is accepted by a NFA.

Let $S(s_i, s_j, k)$ be the set of strings that force M from state s_i to s_j on a path such that every intermediate state is $\{s_0, \ldots, s_k\}$.

- E.g., $S(s_4, s_5, 2)$ consists of all strings that force M from s_4 to s_5 on a path that goes through only s_0, s_1, and s_2 (in any order, perhaps with repeats).

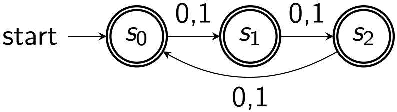
Note that a string x is accepted by M iff $x \in S(s_0, s, n)$ for some final state s. Thus, $L(M)$ is the union over all final states s of $S(s_0, s, n)$.

An example:



$S(s_0, s_1, 0) = \{0, 1\}$; $S(s_0, s_1, 1) = \{0, 1\}$;
$S(s_0, s_1, 2) = \{\text{all strings of length 1 mod 3}\}$.

An example:



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We will prove by induction on k that $S(s_i, s_j, k)$ is regular.

- Why not just take $s_i = s_0$?
  - We need a stronger induction hypothesis

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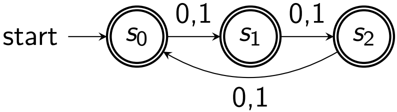

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We will prove by induction on k that $S(s_i, s_j, k)$ is regular.

- Why not just take $s_i = s_0$?
  - We need a stronger induction hypothesis

Base case:

Lemma 1: $S(s_i, s_j, -1)$ is regular.

Proof: For a string σ to be in $S(s_i, s_j, -1)$, it must go directly from s_i to s_j, without going through any intermediate states. Thus, σ must be some subset of I (possibly empty) together with λ if $s_i = s_j$. Either way, $S(s_i, s_j, -1)$ is regular.

Lemma 2: If $s_j \neq s_{k+1}$, then $S(s_i, s_j, k+1) = S(s_i, s_j, k) \cup S(s_i, s_{k+1}, k)(S(s_{k+1}, s_{k+1}, k))^* S(s_{k+1}, s_j, k)$.

Lemma 2: If $s_j \neq s_{k+1}$, then $S(s_i, s_j, k+1) = S(s_i, s_j, k) \cup S(s_i, s_{k+1}, k)(S(s_{k+1}, s_{k+1}, k))^* S(s_{k+1}, s_j, k)$.

Proof: If a string σ forces M from s_i to s_j on a path with intermediates states all in $\{s_0, \ldots, s_{k+1}\}$, then the path either does not go through s_{k+1} at all, so is in $S(s_i, s_j, k)$, or goes through s_{k+1} some finite number of times, say m. That is, the path looks like this:

$$s_i \ldots s_{k+1} \ldots s_{k+1} \ldots s_{k+1} \ldots s_j$$

where all the states in the ... part are in $\{s_0, \ldots, s_k\}$. Thus, we can split up the string σ into $m+1$ corresponding pieces:

- σ_0 that takes M from s_i to s_{k+1},
- each of $\sigma_1, \ldots, \sigma_m$ take M from s_{k+1} back to s_{k+1}
- σ_{m+1} takes M from s_{k+1} to s_j.

Thus,

- $\sigma_0 \in S(s_i, s_{k+1}, k)$
- $\sigma_1, \ldots, \sigma_m$ are all in $S(s_{k+1}, s_{k+1}, k)$
- $\sigma_{m+1} \in S(s_{k+1}, s_j, k)$
- So $\sigma = \sigma_0 \sigma_1 \ldots \sigma_{m+1} \in S(s_i, s_j, k) \cup S(s_i, s_{k+1}, k)(S(s_{k+1}, s_{k+1}, k))^* S(s_{k+1}, s_j, k)$

Lemma 3: If $s_j = s_{k+1}$, then
$S(s_i, s_j, k+1) = S(s_i, s_j, k) \cup S(s_i, s_j, k)(S(s_j, s_j, k))^*$.

Proof: Same idea as previous proof.

Lemma 3: If $s_j = s_{k+1}$, then
$S(s_i, s_j, k+1) = S(s_i, s_j, k) \cup S(s_i, s_j, k)(S(s_j, s_j, k))^*$.

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Proof: Same idea as previous proof.

Lemma 4: $S(s_i, s_j, N)$ is regular for all N with $-1 \leq N \leq n$.

Proof: An easy induction. Lemma 1 gives the base case; Lemmas 2 and 3 give the inductive step.

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$S(s_i, s_j, k+1) = S(s_i, s_j, k) \cup S(s_i, s_j, k)(S(s_j, s_j, k))^*$.

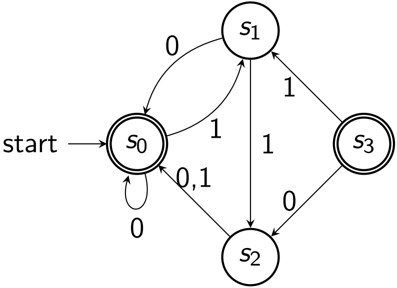
Proof: Same idea as previous proof.

Lemma 4: $S(s_i, s_j, N)$ is regular for all N with $-1 \leq N \leq n$.

Proof: An easy induction. Lemma 1 gives the base case; Lemmas 2 and 3 give the inductive step.

The language accepted by M is the union of the sets $S(s_0, s', n)$ such that s' is a final state. Since regular languages are closed under union, the result follows.

We can use the ideas of this proof to compute the regular language accepted by an automaton.



- $S(s_0, s_0, -1) = \{\lambda, 0\}$; $S(s_0, s_1, -1) = \{1\}$; ...
- $S(s_0, s_0, 0) = 0^*$; $S(s_1, s_0, 0) = 00^*$; $S(s_0, s_1, 0) = 0^*1$; $S(s_1, s_1, 0) = 00^*1$; ...
- $S(s_0, s_0, 1) = (0^*(10)^*)^*$; ...
- ...

We can methodically build up to $S(s_0, s_0, 2)$, which is what we want (since s_3 is unreachable).

A Non-Regular Language

Not every language is regular/accepted by a DFA.

Theorem: $L = \{0^n 1^n : n = 0, 1, 2, \ldots\}$ is not regular.

Proof: Suppose, by way of contradiction, that L is regular. Then there is a DFA $M = (S, \{0, 1\}, f, s_0, F)$ that accepts L. Let $N = |S|$ (i.e., there are N states in M). Let $t_0, \ldots, t_{2N}$ be the path of states that M visits on input $0^N 1^N$

- Thus, $t_0 = s_0$ and t_{2N} is an accepting state
- We must have $f(t_i, 0) = t_{i+1}$ for $i = 0, \ldots, N-1$.

Since M has N states, by the pigeonhole principle, at least two of $t_0, \ldots, t_N$ are the same. Suppose $t_i = t_j$, where $i < j$ and $j - i = d$.

Claim: M accepts $0^N 0^d 1^N$, and $0^N 0^{2d} 1^N$, $0^N 0^{3d} 1^N$, ... So M does not accept L.

Proof: Starting in $t_0 = s_0$, 0^i brings the machine to t_i; another 0^d bring the machine back to t_i (since $t_j = t_{i+d} = t_i$); another 0^d bring machine back to t_i again. After going around the loop for a while, then can continue to t_{2N} and accept.

The Pumping Lemma

The techniques of the previous proof generalize. If M is a DFA and x is a string accepted by M such that $|x| \geq |S|$

- $|S|$ is the number of states; $|x|$ is the length of x

then there are strings u, v, w such that

- $x = uvw$,
- $|uv| \leq |S|$,
- $|v| \geq 1$,
- $uv^i w$ is accepted by M, for $i = 0, 1, 2, \ldots$.

The proof is the same as on the previous slide.

- x was $0^n 1^n$, $u = 0^i$, $v = 0^t$, $w = 0^{N-t-i} 1^N$.

We can use the Pumping Lemma to show that many languages are *not* regular

- $\{1^{n^2} : n = 0, 1, 2, \ldots\}$: homework
- $\{0^{2n} 1^n : n = 0, 1, 2, \ldots\}$: homework
- $\{1^n : n \text{ is prime}\}$
- $\ldots$

More Powerful Machines

Finite automata are very simple machines.

- They have no memory
- Roughly speaking, they can't count beyond the number of states they have.

Pushdown automata have states and a *stack* which provides unlimited memory.

- They can recognize all languages generated by *context-free grammars* (CFGs)
  - CFGs are typically used to characterize the syntax of programming languages
- They can recognize the language $\{0^n 1^n : n = 0, 1, 2, \ldots\}$, but not the language $L' = \{0^n 1^n 2^n : n = 0, 1, 2, \ldots\}$

Linear bounded automata can recognize L'.

- More generally, they can recognize *context-sensitive grammars* (CSGs)
- CSGs are (almost) good enough to characterize the grammar of real languages (like English)

Most general of all: Turing machine (TM)

- Given a *computable* language, there is a TM that accepts it.
- This is essentially how we define computability.

If you're interested in these issues, take CS 4810!