Discrete Structures	Prelim 1
CS2800	

Instructions: This is a 90 minute test. Answer the following questions in the provided booklet. Ensure that your name and netid are on your exam booklet. You may answer the questions in any order, but please mark the questions clearly. Books, notes, calculators, laptops, and carrier pigeons are all disallowed. You may leave mathematical expressions unevaluated (e.g. just write $17 \cdot 3$ instead of 51 and don't bother evaluating $C(17, 3)$). Good luck!

1. [5 points] Recall that $[X \to Y]$ denotes the set of all (total) functions from X to Y, and Y^n denotes $Y \times Y \times \cdots \times Y$ (n times).

   Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set with cardinality n. Define a bijection from $[X \to Y]$ to Y^n.

2. [5 points] Let $\{1, 2, 3\}^\omega$ be the set of infinite sequences containing only the numbers 1, 2, and 3. For example, some sequences of this kind are:

$$(1, 1, 1, 1, 1, 1, \ldots)$$
$$(2, 2, 2, 2, 2, 2, \ldots)$$
$$(3, 2, 1, 3, 2, 1, \ldots)$$

   Prove that $\{1, 2, 3\}^\omega$ is uncountable.

3. [4 points] Prove by induction that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   for all $n \geq 1$.

4. [6 points: 2+4] (a) Explain carefully what the flaw is with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

   We use regular induction to prove that $P(n)$ holds for all $n \geq 3$, where $P(n)$ says that postage of n cents cents can be formed using just three-cent and four-cent stamps.

   Base Case: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

   Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four- cent stamps by three three-cent stamps.

   (b) Show that every postage of six cents or more can be formed using just three-cent and four-cent stamps.

5. [7 points: 4+3] (a) Six women and nine men are on the faculty of a schools CS department. The individuals are distinguishable. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?

   (b) How many ways can Mr. and Mrs. Sweettooth distribute 13 identical pieces of candy to their three children so that each child gets at least one piece of candy?

6. [3 points] Give a combinatorial proof of the following identity:

$$\binom{n}{r+a}\binom{r+a}{r} = \binom{n}{r}\binom{n-r}{a}.$$

   (Note: no credit will be given for an algebraic proof.)