CS 2802: Homework 3

February 8, 2019

Handed out Feb. 4; due Feb. 11

- Read Chapter 5
- Do the following problems:
  - Problem 1: Given a relation R on a set S, recall that for $s \in S$, we can define $[s]_R = \{s' : (s, s') \in R$. That is, $[s]_R$ consists of all the elements in S related to s by R¿ Show that R is an equivalence relation iff (a) the sets $[s]_R$ form a partition of S (i.e., for all $s, s' \in S$, we have either $[s] = [s']$ or $[s] \cap [s']$) and (b) $s \in [s]_R$ for all $s \in S$. (In the last homework, you showed that if R is an equivalence relation, then the sets $[s]_R$ form a partition. You don't have to reprove that. So for this week, you just have to check that if R is an equivalence relation, then $s \in [s]_R$, and that if R has properties (a) and (b) above, then R is an equivalence relation.)
  - Problem 2: Show that if f is a bijection from A to B and g is a bijection from B to C, then $g \circ f$ is a bijection from A to C.
  - Problem 3: Show that if $S \neq \emptyset$, then $f : S \to T$ is an injection iff f has a left inverse.
  - Problem 4: Show that if A and B are countable, then so is $A \cup B$. [You may also want to think about the more general question: How does the cardinality of $A \cup B$ compared to the cardinality of A and B? You don't have to hand this in though.]
  - Problem 5: If $A_0, A_1, A_2, \ldots$ are all countably infinite (it used to say "countable") and disjoint (i.e., $A_i \cap A_j = \emptyset$), construct a bijection between $\cup_{i=0}^{\infty} A_i$ and $\mathbb{N} \times \mathbb{N}$ ($\mathbb{N}$ is the natural numbers). (It used to say $\cup_{i=1}^{\infty} A_i$. I put in a 0 since I started with A_0, although the problem makes sense either way.) Carefully prove that your construction works.
  - Problem 6: (Based on Problem 8.16 in MCS.) In this problem you'll prove what might seem quite surprising: there is a a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$.

(Before it said $(0,\infty]$; now I've corrected it to $[0,\infty)\times[0,\infty)$).) (Some notation: (a,b) denotes the interval $\{x\in\mathbb{R} : a<x<b\}$; similarly, $(a,b]=\{x\in\mathbb{R} : a<x\le b\}$; $[a,b)=\{x\in\mathbb{R} : a\le x<b\}$; $(a,\infty)=\{x\in\mathbb{R} : a<x\}$; and $[a,\infty)=\{x\in\mathbb{R} : a\le x\}$.)

(a) Describe a bijection from $(0,1]$ to $[0,\infty)$. (It used to say $(0,\infty)$. In fact, both versions are correct. There is a bijection between $(0,\infty)$ and $[0,\infty)$, but it's easier to work with $[0,\infty)$.) (Hint: $1/x$ almost works.)

b) An infinite sequence of the decimal digits $\{0,1,\dots,9\}$ will be called *long* if it does not end with all 0s. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all long sequences. Describe a bijection from L to the half-open real interval $(0,\infty)$. (Hint: Put a decimal point at the beginning of the sequence.)

(c) Describe an injection from L to L^2 and an injection from L^2 to L. (For the second injection, consider alternating digits.)

(d) Show that if there is a bijection from A to B, then there is a bijection from $A\times A$ to $B\times B$.

(e) Complete the argument (i.e., show that there is a bijection from $(0,1]$ to $[0,\infty)\times[0,\infty)$) using the Schröder-Bernstein Theorem and Extra Problem 4 from HW2.

Note that the parts of this problem are independent. For example, you can do (d) even if you don't do (a), (b), and (c). When you're doing a later part, you can use the earlier parts, even if you didn't prove them.

– Problem 7: Give a bijection between infinite binary sequences (i.e., infinite sequences of 0s and 1s) and subsets of the natural numbers.

Something to think about (but not to hand in:) Prove there is a bijection between the real numbers in $[0,1]$ and subsets of the natural numbers. (Hint: every real number in $[0,1]$ can be written as a binary decimal, and a binary decimal determines an infinite binary sequence. The only slight problem is that a binary decimal ending in all 1s, like $.01111\dots$, is equivalent to another binary decimal ending in all 0s; for example, $.01111\dots = .10000\dots$. This is why we considered long sequences in the Problem 6. But it's not too hard to show that the set of long binary sequences (i.e., the binary sequences that don't end in an infinite sequence of 0s) has the same cardinality as the set of all binary sequences.)