Discrete Structures	Prelim 2 sample questions
CS2800	Questions selected for fall 2017

1 Functions, relations, and infinite cardinality

1. True/false. For each of the following statements, indicate whether the statement is true or false. Give a one or two sentence explanation for your answer.

   (a) The relation $\leq$ is an equivalence relation

   (b) The set of real numbers ($\mathbb{R}$) is countable.

   (c) The set of rational numbers ($\mathbb{Q}$) is countable.

   (d) If there is a bijection from $\mathbb{Q}$ to X then X is countable.

   (e) Recall that $[X \to Y]$ denotes the set of functions with domain X and codomain Y. Let $f : 2^S \to [S \to \{0,1\}]$ be given by $f(X) ::= h$ where $h : S \to \{0,1\}$ is given by $h(s) ::= 0$. f is injective.

   (f) f as just defined is surjective.

   (g) If a function has a right inverse, then the right inverse is unique.

2. Complete the following diagonalization proof:

   Claim: $X = [\mathbb{N} \to \mathbb{N}]$ is uncountable.

   Proof: We prove this claim by contradiction. Assume that X is countable. Then there exists a function F : **FILL IN** that is **FILL IN**.

   Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of X in a table:

	0	1	2	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

   Let f_D : **FILL IN** be given by $f_D : x \mapsto$ **FILL IN**

   Then **FILL IN**

   This is a contradiction because **FILL IN**.

3. Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.

   (a) The set Σ^* containing all finite length strings of 0's and 1's.

   (b) The set $2^{\mathbb{N}}$ containing all sets of natural numbers.

   (c) The set $\mathbb{N} \times \mathbb{N}$ containing all pairs of natural numbers.

   (d) The set $[\mathbb{N} \to \{0,1\}]$ containing all functions from $\mathbb{N}$ to $\{0,1\}$.

   Be sure to include enough detail:

   - If listing elements, be sure to clearly state how you are listing them;
   - If diagonalizing, be sure it is clear what your diagonal construction is;
   - If providing a function, make sure it is clear what the output is on a given input.

4. For any function $f : A \to B$ and a set $C \subseteq A$, define $f(C) = \{f(x) \mid x \in C\}$. That is, $f(C)$ is the set of images of elements of C. Prove that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.

   (*Hint:* one way to prove this is from the definition of set equality: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.)

5. (a) Write the definition of "$f : A \to B$ is injective" using formal notation ($\forall$, $\exists$, "and", "or", "if ... then ...", $=$, $\neq$, ...).

   (b) Similarly, write down the definition of "$f : A \to B$ is surjective".

   (c) Write down the definition of "A is countable". You may write "f is surjective" or "f is injective" in your expression.

6. Recall that the composition of two functions $f : B \to C$ and $g : A \to B$ is the function $f \circ g : A \to C$ defined as $(f \circ g)(x) = f(g(x))$. Prove that if f and g are both injective, then $f \circ g$ is injective.

7. For each of the following functions, indicate whether the function f is injective, whether it is surjective, and whether it is bijective. Give a one sentence explanation for each answer.

   (a) $f : \mathbb{N} \to \mathbb{N}$ given by $f : x \to x^2$

   (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \to x^2$

   (c) $f : X \to [Y \to X]$ given by $f(x) := h_x$ where $h_x : Y \to X$ is given by $h_x(y) := x$.

8. [6 points] Recall that $[X \to Y]$ denotes the set of functions with domain X and codomain Y. Let X and Y be nonempty sets, and let $F : [X \to Y] \to [X \to (Y \times Y)]$ be given by $F(f) ::= h_f$, where $h_f : X \to (Y \times Y)$ is given by $h_f(x) ::= (f(x), f(x))$ for all x.

   (a) Show that F is injective. *Note:* $g_1 = g_2$ if and only if, for all x, $g_1(x) = g_2(x)$.

   (b) Show that F is not necessarily bijective.

2 Combinatorics

1. Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.

   (a) During your pregnancy, you decided on a list of 23 girls' first names and 16 boys' first names, as well as a list of 11 gender-neutral middle names. To your surprise, you had quintuplets, two boys and three girls. Now you must select a first and a middle name for each child from the lists. The names must all be different.

   (b) A professor teaching discrete structures is making up a final exam. He has a stash of 24 questions on probability, 16 questions on combinatorics, and 10 questions on logic. He wishes to put five questions on each topic on the exam.

   (c) The very same professor wants to assign points to the 15 problems so that each problem is worth at least 5 points and the total number of points is 100.

   (d) There are 30 graders to grade the final exam, and the professor would like to assign two graders to each of the 15 problems.

2. Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.

(a) You must choose a password consisting of 6, 7, or 8 letters from the 26-letter English alphabet $\{a, b, \ldots, z\}$.

(b) In a poker game, you are dealt a *full house*, a five-card hand containing three of a kind and a pair of another kind; for example, three kings and two sixes.

(c) Your team is in the championship game of a soccer tournament. The score is tied at full time and the winner will be decided by penalty kicks. As coach, you must choose a sequence of five different players out of 11 to take the kicks.

(d) You have ten rings, all with different gemstones. You wish to bequeath them to your five children so that each child inherits two of the rings.

(e) You have \$400 to donate to charity, which you would like to distribute among your five favorite charities so that each receives an integral number of dollars.

3. [8 points]

   (a) Let A be the set of permutations of the string $JUICE$. What is $|A|$?

   (b) Define a relation $\sim$ on A by $x \sim y$ if we can rearrange the vowels of x or the consonants of x (or both) to form y. For example, $JUICE \sim CEIJU$ but $JUICE \not\sim JUCIE$. List 4 of the elements of $[JUICE]_\sim$.

   (c) How many elements are there in each equivalence class? Briefly explain.

   (d) What is $|A/\sim|$?

   (e) Let $f : (A/\sim) \to \{J, U, I, C, E\}$ be given by $f([x]_\sim) ::=$ the first letter of x. Is f a well-defined function? Briefly explain.

3 Induction

1. [6 points] *Pascal's triangle* is a sequence of rows, where each entry is formed by adding the two adjacent entries from the previous row:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \cdots & & & & \end{array}$$

   If we let $P_{n,k}$ stand for the kth entry in the nth row of Pascal's triangle, then $P_{n,k}$ is given by the formulas $P_{1,1} ::= 1$, $P_{n,0} ::= 0$ for all n, and $P_{n+1,k} ::= P_{n,k-1} + P_{n,k}$ if $n \geq 1$.

   Prove by induction on n that for all $n \geq 1$, for all k with $1 \leq k \leq n$, $P_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

   Note: The definition of $n!$ is $0! ::= 1$ and $n! ::= n \cdot (n-1)!$ for all $n \geq 1$.

2. Prove the following claim using induction: for any $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

3. We define a set S of functions from $\mathbb{Z}$ to $\mathbb{Z}$ inductively as follows:

   Rule 1. For any $n \in \mathbb{Z}$, the translation (or offset) function $t_n : x \mapsto x + n$ is in S.

Rule 2. For any $k \neq 0 \in \mathbb{Z}$, the scaling function $r_k : x \mapsto kx$ is in S.

Rule 3. If f and g are elements of S, then the composition $f \circ g \in S$.

Rule 4. If $f \in S$ and f has a right inverse g, then g is also in S.

In other words, S consists of functions that translate and scale integers, and compositions and right inverses thereof.

> **Note:** This semester, we made a bigger distinction between the elements of an inductively defined set and the meaning of an inductively defined set. We probably would have phrased this question as follows: Let S be given by
>
> $$s \in S ::= t_n \mid r_k \mid s_1 \circ s_2 \mid rinv\ s$$
>
> and inductively, let the function defined by s (written $F_s : \mathbb{Z} \to \mathbb{Z}$) be given by the rules $F_{t_n}(x) ::= x + n$, $F_{r_k}(x) ::= ks$, $F_{s_1 \circ s_2}(x) ::= F_{s_1} \circ F_{s_2}$ and let $F_{rinv\ s} ::= g$ where g is a right inverse of F_s.

(a) [1 point] Show that the function $f : x \mapsto 3x + 17$ is in S.

(b) Use structural induction to prove that for all $f \in S$, f is injective. You may use without proof the fact that the composition of injective functions is injective.

(c) Give a surjection ϕ from S to $\mathbb{Z}$ (proof of surjectivity not necessary). Remember that this surjection must map a *function* to an *integer*, and for every integer there must be a function that maps to it.

4. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers n (including 0):

$$\sum_{i=0}^{n} F_i \;=\; F_{n+2} - 1$$

5. Prove by induction that for any integer $n \geq 3$, $n^2 - 7n + 12$ is non-negative.

4 Automata

1. Given DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$, we can construct a machine M_{12} with $L(M_{12}) = L(M_1) \cap L(M_2)$ as follows:

- Let $Q = Q_1 \times Q_2 =$ the set of all ordered pairs (q_1, q_2), where $q_1 \in Q_1$ and $q_2 \in Q_2$.
- Let $q_0 \in Q = (q_{01}, q_{02})$.
- Let $F = F_1 \times F_2 = \{(q_1, q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2\}$.
- Let $\delta_{12}((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.
- Let $M_{12} = (Q, \Sigma, \delta_{12}, q_0, F)$.

Use structural induction to prove that for all $x \in \Sigma^*$, $\widehat{\delta}_{12}((q_1, q_2), x) = \left(\widehat{\delta}_1(q_1, x), \widehat{\delta}_2(q_2, x)\right)$.

2. Draw a finite automaton (DFA, NFA or ϵ-NFA) with alphabet $\{0, 1\}$ to recognize the language

$$\{x \in \{0, 1\}^* \mid x \text{ contains the substring } 010\}$$

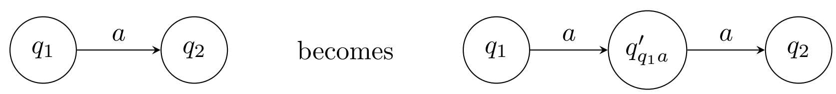
3. Draw a finite automaton (DFA, NFA or ϵ-NFA) with alphabet $\{a, b\}$ to recognize strings of the form $x_1x_2x_3\cdots$ where each x_i is either "ab" or "ba".

4. Prove that $L = \{0^n10^n \mid n \in \mathbb{N}\}$ is not DFA-recognizable.

5. Build a deterministic finite automaton that recognizes the set of strings of 0's and 1's, that only contain a single 0 (and any number of 1's). Describe the set of strings that lead to each state.

6. Given a string x, we can define the "character doubling" of x to be x with every character doubled: for example $cd(abc) = aabbcc$. Formally, $cd(\epsilon) = \epsilon$, and $cd(xa) = cd(x)aa$. We can then define the "character doubling" of a language L to be the set of all strings formed by doubling the characters of strings in L; formally $cd(L) = \{cd(x) \mid x \in L\}$.

   Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, we can construct a new DFA M_{cd} that recognizes $cd(L(M))$ by adding a new state q'_{qa} to the middle of every transition from q on character a:



   (a) Formally describe the components $(Q_{cd}, \Sigma_{cd}, \delta_{cd}, q_{0cd}, F_{cd})$ of M_{cd} in terms of the components of M. Be sure to describe δ_{cd} on all inputs (you may need to add one or more additional states).

   (b) Use structural induction on x to prove that for all x, $\widehat{\delta}(q_0, x) = \widehat{\delta}_{cd}(q_{0cd}, cd(x))$.

   (c) We can also define the "string doubling" of x to be xx. For example, $sd(abc) = abcabc$. Show that the set of regular languages is *not* closed under string doubling. In other words, give a regular language L and prove that $sd(L) = \{sd(x) \mid x \in L\}$ is not regular.

   You can use any theorem proved in class to help prove this result.

7. Happy Cat has been shown the following proof, and has promptly turned into Grumpy Cat. Briefly but clearly identify the error which has induced grumpiness.

   To prove: The language of the regular expression 0^*1^* is, in fact, not DFA-recognizable.

   Proof. Let L be the language of 0^*1^*. Assume there is some DFA M with n states that recognizes L. Let $x = 0^{n-1}11$. Clearly, $x \in L$ and $|x| \geq n$. Therefore according to the Pumping Lemma, we can split x into three parts u, v and w, such that $|uv| \leq n$, $|v| \geq 1$, and $uv^iw \in L$ for all natural numbers i. Let $|v| = n$. Since $|uv| \leq n$, it must be the case that $u = \epsilon$, and $v = 0^{n-1}1$. Then $uv^2w = 0^{n-1}10^{n-1}11$, which is clearly not in L. This contradicts our assumption that there is a DFA which recognizes the language.