Discrete Structures	*Prelim 2 sample questions— **Solutions***
CS2800	*Questions selected for fall 2017*

1 Functions, relations, and infinite cardinality

1. *True/false. For each of the following statements, indicate whether the statement is true or false. Give a one or two sentence explanation for your answer.*

   (a) *The relation $\leq$ is an equivalence relation*

   Solution False. It is not symmetric, because (for example) $1 \leq 2$ but $2 \not\leq 1$.

   (b) *The set of real numbers ($\mathbb{R}$) is countable.*

   Solution **False.** We proved this in class using diagonalization.

   (c) *The set of rational numbers ($\mathbb{Q}$) is countable.*

   Solution **True.** We proved this in class by giving a procedure for listing all of the rational numbers (by putting them in a table and traversing the diagonals of the table).

   (d) *If there is a bijection from $\mathbb{Q}$ to X then X is countable.*

   Solution True. We know that $|\mathbb{Q}| = |\mathbb{N}|$. If there is a bijection from $\mathbb{Q} \to X$, then $|\mathbb{Q}| = |X|$. This means $|X| = |\mathbb{N}|$, so X is countable.

   (e) *Recall that $[X \to Y]$ denotes the set of functions with domain X and codomain Y. Let $f : 2^S \to [S \to \{0,1\}]$ be given by $f(X) ::= h$ where $h : S \to \{0,1\}$ is given by $h(s) ::= 0$. f is injective.*

   Solution **False.** f always returns the same thing, so it can't be one to one. For example, choose any two different subsets X_1 and X_2 of S; then $f(X_1) = h = f(X_2)$.

   (f) *f as just defined is surjective.*

   Solution **False.** Choose any function $h' : S \to \{0,1\}$ other than h. Since f only outputs h, it never outputs h'.

   (g) *If a function has a right inverse, then the right inverse is unique.*

   Solution False. Let $f : \{0,1,2\} \to \{a,b\}$ be given by $f(0) ::= a$, $f(1) ::= a$ and $f(2) ::= b$. Then $g_1 : \{a,b\}$ given by $g_1(a) ::= 0$ and $g_1(b) ::= 2$ is a right inverse, but so is g_2 given by $g_2(a) ::= 1$ and $g_2(b) ::= 2$.

2. *Complete the following diagonalization proof:*

   Claim: *$X = [\mathbb{N} \to \mathbb{N}]$ is uncountable.*

   Proof: *We prove this claim by contradiction. Assume that X is countable. Then there exists a function F : **FILL IN** that is **FILL IN**.*

   Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of X in a table:

	0	*1*	*2*	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

Let f_D : ***FILL IN*** *be given by $f_D : x \mapsto$* ***FILL IN***

Then ***FILL IN***

This is a contradiction because ***FILL IN****.*

Solution **Claim:** $X = [\mathbb{N} \to \mathbb{N}]$ is uncountable.

Proof: We prove this claim by contradiction. Assume that X is countable. Then there exists a function $F : \mathbb{N} \to X$ that is **surjective**.

Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of X in a table:

	0	1	2	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

Let $f_D : \mathbb{N} \to \mathbb{N}$ be given by $f_D : x \mapsto 1 + f_x(x)$

Then f_D **is not in the table, because for any i, it differs from f_i on input i.**

This is a contradiction because **we assumed F was surjective**.

3. *Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.*

   (a) *The set Σ^* containing all finite length strings of 0's and 1's.*

   Solution This set is countable. You can list all strings of length 0, then all strings of length one, then all strings of length 2, and so on.

   (b) *The set $2^{\mathbb{N}}$ containing all sets of natural numbers.*

   Solution This set is not countable. If it were, we could put all of the sets in a table:

	0	1	2	$\cdots$
S_1	$0 \in S_1$	$\notin$	$\in$	$\cdots$
S_2	$\notin$	$\notin$	$\notin$	$\cdots$
S_3	$\in$	$\notin$	$\in$	$\cdots$

   We can then construct the set S_D by swapping everything on the diagonal ($S_D = \{i \mid i \notin S_i\}$). Then $S_D \neq S_k$ for any k, because $k \in S_D$ if and only if $k \notin S_k$. Thus S_D is not in the table, which contradicts the fact that the table contained all sets.

   (c) *The set $\mathbb{N} \times \mathbb{N}$ containing all pairs of natural numbers.*

   Solution This set is countable. You can put all of the pairs in a table, and then map the natural numbers to the pairs by tracing diagonals of the table.

   (d) *The set $[\mathbb{N} \to \{0, 1\}]$ containing all functions from $\mathbb{N}$ to $\{0, 1\}$.*

Solution This set is not countable. There is a bijection between $[\mathbb{N} \to \{0,1\}]$ and $2^{\mathbb{N}}$, and we showed above that $2^{\mathbb{N}}$ is uncountable. Alternatively, you can diagonalize directly using the function $f : n \mapsto f_n(n) + 1$ or similar.

Be sure to include enough detail:

- *If listing elements, be sure to clearly state how you are listing them;*
- *If diagonalizing, be sure it is clear what your diagonal construction is;*
- *If providing a function, make sure it is clear what the output is on a given input.*

4. *For any function $f : A \to B$ and a set $C \subseteq A$, define $f(C) = \{f(x) \mid x \in C\}$. That is, $f(C)$ is the set of images of elements of C. Prove that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.*

   *(*Hint: *one way to prove this is from the definition of set equality: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.)*

   Solution Choose an arbitrary $b \in f(C_1 \cap C_2)$. We wish to show $x \in f(C_1) \cap f(C_2)$. Since $b \in f(C_1 \cap C_2)$, there must exist some $a \in C_1 \cap C_2$ with $f(a) = b$. Since $a \in C_1 \cap C_2$, we have $a \in C_1$ so $b = f(a) \in f(C_1)$; similarly, $b \in f(C_2)$. Therefore, $b \in f(C_1) \cap f(C_2)$.

   Conversely, choose an arbitrary $b \in f(C_1) \cap f(C_2)$. We want to show $b \in f(C_1 \cap C_2)$. Now, $b = f(a_1)$ for some $a_1 \in C_1$, and $b = f(a_2)$ for some $a_2 \in C_2$. Since f is injective, $a_1 = a_2$, so a_1 is also in C_2. Therefore, $a_1 \in C_1 \cap C_2$ so $b = f(a_1) \in f(C_1 \cap C_2)$ as required.

5. (a) *Write the definition of "$f : A \to B$ is injective" using formal notation ($\forall$, $\exists$, "and", "or", "if ... then ...", $=$, $\neq$, ...).*

   Solution $\forall x_1, x_2 \in A$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$

   (b) *Similarly, write down the definition of "$f : A \to B$ is surjective".*

   Solution $\forall b \in B, \exists a \in A, f(a) = b$.

   (c) *Write down the definition of "A is countable". You may write "f is surjective" or "f is injective" in your expression.*

   Solution $\exists f : \mathbb{N} \to A$ such that f is surjective.

6. *Recall that the composition of two functions $f : B \to C$ and $g : A \to B$ is the function $f \circ g : A \to C$ defined as $(f \circ g)(x) = f(g(x))$. Prove that if f and g are both injective, then $f \circ g$ is injective.*

   Solution Assume that f and g are injective, and assume that $(f \circ g)(a_1) = (f \circ g)(a_2)$. By definition, we have $f(g(a_1)) = f(g(a_2))$. Since f is injective, we conclude $g(a_1) = g(a_2)$; since g is injective, we conclude $a_1 = a_2$.

7. *For each of the following functions, indicate whether the function f is injective, whether it is surjective, and whether it is bijective. Give a one sentence explanation for each answer.*

   (a) *$f : \mathbb{N} \to \mathbb{N}$ given by $f : x \to x^2$*

   Solution f is injective because if $x^2 = y^2$ then $x = \pm y$ but the domain is $\mathbb{N}$ so x cannot be $-y$.

   f is not surjective, for example there is no $x \in \mathbb{N}$ with $x^2 = 2$.

   f is not bijective because it is not surjective.

   (b) *$f : \mathbb{R} \to \mathbb{R}$ given by $f : x \to x^2$*

Solution f is not injective; for example $f(2) = f(-2)$.

f is not surjective; for example there is no $x \in \mathbb{R}$ with $x^2 = -1$.

f is not bijective because it is not injective.

(c) *$f : X \to [Y \to X]$ given by $f(x) := h_x$ where $h_x : Y \to X$ is given by $h_x(y) := x$.*

Solution f is injective if Y is nonempty. If $f(x) = f(x')$ then for all y, $h_x(y) = h_{x'}(y)$. But if Y is non-empty, then there is some $y \in Y$, so $x = h_x(y) = h_{x'}(y) = x'$.

f is not surjective. For example, if $X = Y = \mathbb{R}$, the function $h(x) := x^2$ is not in the image of f.

f is not bijective because it is not surjective.

8. *[6 points] Recall that $[X \to Y]$ denotes the set of functions with domain X and codomain Y. Let X and Y be nonempty sets, and let $F : [X \to Y] \to [X \to (Y \times Y)]$ be given by $F(f) ::= h_f$, where $h_f : X \to (Y \times Y)$ is given by $h_f(x) ::= (f(x), f(x))$ for all x.*

   (a) *Show that F is injective.* Note: $g_1 = g_2$ *if and only if, for all x, $g_1(x) = g_2(x)$.*

   Solution Assume $F(f_1) = F(f_2)$. Then $h_{f_1}(x) = h_{f_2}(x)$ for all x. That means that for all x, $(f_1(x), f_1(x)) = (f_2(x), f_2(x))$. This in turn implies that $f_1(x) = f_2(x)$. Since this is true for all x, $f_1 = f_2$, as required.

   (b) *Show that F is not necessarily bijective.*

   Solution F only outputs functions that output pairs with the same first and second components. Any function that outputs a pair with different first and second component will not be in the image of F.

   For example, let $X = \{0, 1\}$ and $Y = \{a, b\}$, and let $g : X \to Y \times Y$ be given by $g(x) = (a, b)$. Then g is not in the image of F.

2 Combinatorics

1. *Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.*

   (a) *During your pregnancy, you decided on a list of 23 girls' first names and 16 boys' first names, as well as a list of 11 gender-neutral middle names. To your surprise, you had quintuplets, two boys and three girls. Now you must select a first and a middle name for each child from the lists. The names must all be different.*

   Solution $P(16, 2) \cdot P(23, 3) \cdot P(11, 5)$ or equivalent, e.g.
   $(16!/14!)(23!/20!)(11!/6!)$, $16 \cdot 15 \cdot 23 \cdot 22 \cdot 21 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

   (b) *A professor teaching discrete structures is making up a final exam. He has a stash of 24 questions on probability, 16 questions on combinatorics, and 10 questions on logic. He wishes to put five questions on each topic on the exam.*

Solution $\binom{24}{5}\binom{16}{5}\binom{10}{5}$ or equivalent

(c) *The very same professor wants to assign points to the 15 problems so that each problem is worth at least 5 points and the total number of points is 100.*

Solution $\binom{25+15-1}{25}$ or $\binom{25+15-1}{15-1}$ or equivalent

(d) *There are 30 graders to grade the final exam, and the professor would like to assign two graders to each of the 15 problems.*

Solution $\binom{30}{2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2}$ or $\frac{30!}{2^{15}}$ or $\binom{30}{2}\binom{28}{2}\binom{26}{2}\cdots\binom{2}{2}$

2. *Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.*

(a) *You must choose a password consisting of 6, 7, or 8 letters from the 26-letter English alphabet* $\{a, b, \ldots, z\}$.

Solution $26^6 + 26^7 + 26^8$

(b) *In a poker game, you are dealt a* full house, *a five-card hand containing three of a kind and a pair of another kind; for example, three kings and two sixes.*

Solution $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$

(c) *Your team is in the championship game of a soccer tournament. The score is tied at full time and the winner will be decided by penalty kicks. As coach, you must choose a sequence of five different players out of 11 to take the kicks.*

Solution $P(11, 5) = 11!/(11-5)! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

(d) *You have ten rings, all with different gemstones. You wish to bequeath them to your five children so that each child inherits two of the rings.*

Solution $\binom{10}{2\ 2\ 2\ 2\ 2} = \frac{10!}{2!\,2!\,2!\,2!\,2!}$

(e) *You have $400 to donate to charity, which you would like to distribute among your five favorite charities so that each receives an integral number of dollars.*

Solution $\binom{400+5-1}{400}$

3. *[8 points]*

   (a) *Let A be the set of permutations of the string $JUICE$. What is $|A|$?*

   Solution 5!.

   (b) *Define a relation $\sim$ on A by $x \sim y$ if we can rearrange the vowels of x or the consonants of x (or both) to form y. For example, $JUICE \sim CEIJU$ but $JUICE \not\sim JUCIE$. List 4 of the elements of $[JUICE]_\sim$.*

   Solution $JUICE$, $CUIJE$, $CEIJU$, $CIEJU$.

   (c) *How many elements are there in each equivalence class? Briefly explain.*

   Solution Given a string, we can form an equivalent string by choosing a permutation of the vowels and a permutation of the consonants. There are 3! permutations of the vowels, and 2! permutations of the consonants, so there are 3!·2! equivalent strings, so each equivalence class contains 3!2! strings.

   (d) *What is $|A/\sim|$?*

   Solution 5!/2!3! by the division rule.

   (e) *Let $f : (A/\sim) \to \{J, U, I, C, E\}$ be given by $f([x]_\sim)$ ::= the first letter of x. Is f a well-defined function? Briefly explain.*

   Solution No. $JUICE = CUIJE$ but $f([JUICE]) = J \neq C = f([CUIJE])$.

3 Induction

1. *[6 points]* Pascal's triangle *is a sequence of rows, where each entry is formed by adding the two adjacent entries from the previous row:*

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ \cdots \end{array}$$

   If we let $P_{n,k}$ stand for the kth entry in the nth row of Pascal's triangle, then $P_{n,k}$ is given by the formulas $P_{1,1} ::= 1$, $P_{n,0} ::= 0$ for all n, and $P_{n+1,k} ::= P_{n,k-1} + P_{n,k}$ if $n \geq 1$.

   Prove by induction on n that for all $n \geq 1$, for all k with $1 \leq k \leq n$, $P_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

   Note: *The definition of $n!$ is $0! ::= 1$ and $n! ::= n \cdot (n-1)!$ for all $n \geq 1$.*

Solution Proof by induction. Let $P(n)$ be the statement $P_{n,k} = \binom{n}{k}$.

$P(1)$ is true, because $P_{1,1} = 1$ and $\binom{1}{1} = 1!/0!1! = 1$.

Now, assume $P(n)$; we wish to show $P(n+1)$. Well,

$$\begin{aligned}
P_{n+1,k} &= P_{n,k-1} + P_{n,k} && \text{by definition} \\
&= \binom{n}{k-1} + \binom{n}{k} && \text{by induction hypothesis} \\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} && \text{by definition} \\
&= n!\frac{k + (n-k+1)}{k!(n-k+1)!} && \text{putting things over a common denominator} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} && \text{algebra} \\
&= \binom{n+1}{k} && \text{by definition}
\end{aligned}$$

as required.

2. *Prove the following claim using induction: for any $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$*

Solution Base case: when $n = 0$, the left hand side is $2^0 = 1$ and the right hand side is $2^2 - 1 = 1$, and they are clearly the same.

Inductive step: Choose an arbitrary n and assume that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ (this is the inductive hypothesis).

We wish to show that $\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$. We compute:

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} && \text{arithmetic} \\
&= (2^{n+1} - 1) + 2^{n+1} && \text{by the inductive hypothesis} \\
&= 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1
\end{aligned}$$

as required.

3. *We define a set S of functions from $\mathbb{Z}$ to $\mathbb{Z}$ inductively as follows:*

Rule 1. *For any $n \in \mathbb{Z}$, the translation (or offset) function $t_n : x \mapsto x + n$ is in S.*

Rule 2. *For any $k \neq 0 \in \mathbb{Z}$, the scaling function $r_k : x \mapsto kx$ is in S.*

Rule 3. *If f and g are elements of S, then the composition $f \circ g \in S$.*

Rule 4. *If $f \in S$ and f has a right inverse g, then g is also in S.*

In other words, S consists of functions that translate and scale integers, and compositions and right inverses thereof.

> ***Note:*** *This semester, we made a bigger distinction between the elements of an inductively defined set and the meaning of an inductively defined set. We probably would have phrased this question as follows: Let S be given by*
>
> $$s \in S ::= t_n \mid r_k \mid s_1 \circ s_2 \mid rinv\ s$$

and inductively, let the function defined by s (written $F_s : \mathbb{Z} \to \mathbb{Z}$) be given by the rules $F_{t_n}(x) ::= x + n$, $F_{r_k}(x) ::= ks$, $F_{s_1 \circ s_2}(x) ::= F_{s_1} \circ F_{s_2}$ and let $F_{rinv\ s} ::= g$ where g is a right inverse of F_s.

(a) *[1 point] Show that the function $f : x \mapsto 3x + 17$ is in S.*

Solution By rule 1, the function $t_{17} : x \mapsto x + 17$ is in S, and by rule 2, $r_3 : x \mapsto 3x$ is in S. By rule 3, therefore, $t_{17} \circ r_3 : x \mapsto 3x + 17$ is in S.

(b) *Use structural induction to prove that for all $f \in S$, f is injective. You may use without proof the fact that the composition of injective functions is injective.*

Solution We must show that all functions formed with each of the rules are injective. Let $P(s)$ be the statement s is injective.

$P(t_k)$ holds, because t_k has a two sided inverse t_{-k}, and is therefore injective.

$P(r_k)$ holds, because we required that $k \neq 0$. Therefore, if $kx_1 = kx_2$, we can cancel k to find $x_1 = x_2$.

$P(f \circ g)$ holds, assuming $P(f)$ and $P(g)$, because the composition of injections is an injection.

If g is the right inverse of f, then $P(g)$ holds, because g has a left-inverse (namely f) and is therefore injective.

(c) *Give a surjection ϕ from S to $\mathbb{Z}$ (proof of surjectivity not necessary). Remember that this surjection must map a* function *to an* integer*, and for every integer there must be a function that maps to it.*

Solution Let $\phi(s) ::= s(0)$. This is a surjection, because $t_n(0) = 0 + n = n$, so for any n there exists $s \in S$ (namely t_n) with $\phi(s) = n$.

4. *The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:*

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \textit{for } n \geq 2 \end{aligned}$$

That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers n (including 0):

$$\sum_{i=0}^{n} F_i \;=\; F_{n+2} - 1$$

Solution Let $P(n)$ be the statement "$\sum_{i=0}^{n} F_i = F_{n+2} - 1$. We must show $P(0)$ and $P(n+1)$ assuming $P(n)$.

To see $P(0)$, note that $\sum_{i=0}^{0} F_i = F_0 = 1$, while $F_{0+2} - 1 = F_0 + F_1 - 1 = 1 + 1 - 1 = 1$. Since they are the same, $P(0)$ holds.

To see $P(n+1)$, first assume $P(n)$. We have

$$\begin{aligned} \sum_{i=0}^{n+1} F_i &= \sum_{i=0}^{n} F_i + F_{n+1} \\ &= F_{n+2} - 1 + F_{n+1} && \text{by } P(n) \\ &= F_{n+1+2} - 1 && \text{by definition of } F_{n+1+2} \end{aligned}$$

as required.

5. *Prove by induction that for any integer $n \geq 3$, $n^2 - 7n + 12$ is non-negative.*

Solution Let $P(n)$ be the statement "$n^2 - 7n + 12$ is non-negative." We must show $P(3)$, and for any $n \geq 3$, $P(n+1)$ assuming $P(n)$.

To see $P(3)$, note that $3^2 - 7 \cdot 3 + 12 = 0 \geq 0$.

Now, assume $n \geq 3$ and $P(n)$; we want to show $P(n+1)$. Well,

$$\begin{aligned}(n+1)^2 - 7(n+1) + 12 &= n^2 + 2n + 1 - 7n - 7 + 12 \\ &= (n^2 - 7n + 12) + (2n + 1 - 7) \\ &\geq 2n + 1 - 7 && \text{by } P(n) \\ &\geq 0 && \text{since } n \geq 3\end{aligned}$$

4 Automata

1. *Given DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$, we can construct a machine M_{12} with $L(M_{12}) = L(M_1) \cap L(M_2)$ as follows:*
   - *Let $Q = Q_1 \times Q_2 =$ the set of all ordered pairs (q_1, q_2), where $q_1 \in Q_1$ and $q_2 \in Q_2$.*
   - *Let $q_0 \in Q = (q_{01}, q_{02})$.*
   - *Let $F = F_1 \times F_2 = \{(q_1, q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2\}$.*
   - *Let $\delta_{12}((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.*
   - *Let $M_{12} = (Q, \Sigma, \delta_{12}, q_0, F)$.*

   Use structural induction to prove that for all $x \in \Sigma^*$, $\widehat{\delta}_{12}((q_1, q_2), x) = \Big(\widehat{\delta}_1(q_1, x), \widehat{\delta}_2(q_2, x)\Big)$.

Solution The main challenge here is wading through the notational jungle to understand what the problem actually says. Once you've done this, the proof is short and straightforward. Here it is in all its glory:

> We will prove the result by structural induction on x, as suggested. Both the set of strings Σ^* and the extended transition function $\widehat{\delta}$ are defined recursively (see the definitions at the end). The base case for x is the empty string ϵ. We can simply read off the corresponding line in the definition of $\widehat{\delta}$, which tells us that
>
> - $\widehat{\delta}_1(q_1, \epsilon) = q_1$,
> - $\widehat{\delta}_2(q_2, \epsilon) = q_2$, and
> - $\widehat{\delta}_{12}((q_1, q_2), \epsilon) = (q_1, q_2)$.
>
> Hence $\widehat{\delta}_{12}((q_1, q_2), \epsilon) = (q_1, q_2) = \Big(\widehat{\delta}_1(q_1, \epsilon), \widehat{\delta}_2(q_2, \epsilon)\Big)$, so the statement is true in the base case.
>
> Now assume the statement is true for some string x, and consider the "next larger" string xa.
>
> Again reading off the appropriate line in the definition of $\widehat{\delta}$, we know that
>
> - $\widehat{\delta}_1(q_1, xa) = \delta_1(\widehat{\delta}_1(q_1, x), a)$, and
> - $\widehat{\delta}_2(q_2, xa) = \delta_2(\widehat{\delta}_2(q_2, x), a)$

What is $\widehat{\delta}_{12}((q_1, q_2), xa)$? Well, we also have

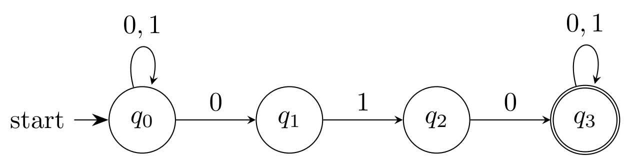
$$\begin{aligned}
\widehat{\delta}_{12}((q_1, q_2), xa) &= \delta_{12}(\widehat{\delta}_{12}((q_1, q_2), x), a) && \text{(definition of } \widehat{\delta}_{12}\text{)} \\
&= \delta_{12}\left(\left(\widehat{\delta}_1(q_1, x), \widehat{\delta}_2(q_2, x)\right), a\right) && \text{(inductive hypothesis)} \\
&= \left(\delta_1\left(\widehat{\delta}_1(q_1, x), a\right), \delta_2\left(\widehat{\delta}_2(q_2, x), a\right)\right) && \text{(definition of } \delta_{12}\text{)} \\
&= \left(\widehat{\delta}_1(q_1, xa), \widehat{\delta}_2(q_2, xa)\right) && \text{(definition of } \widehat{\delta}_1, \widehat{\delta}_2\text{)}
\end{aligned}$$

This proves the statement for all strings $x \in \Sigma^*$ by (structural) induction.

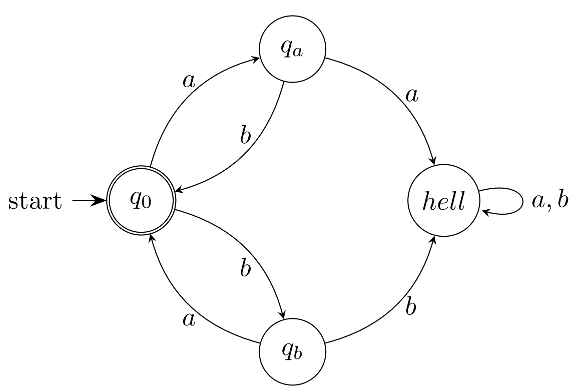
2. *Draw a finite automaton (DFA, NFA or ϵ-NFA) with alphabet $\{0, 1\}$ to recognize the language*

$$\{x \in \{0,1\}^* \mid x \textit{ contains the substring } 010\}$$

Solution An NFA is probably the easiest to construct.



3. *Draw a finite automaton (DFA, NFA or ϵ-NFA) with alphabet $\{a, b\}$ to recognize strings of the form $x_1x_2x_3\cdots$ where each x_i is either "ab" or "ba".*



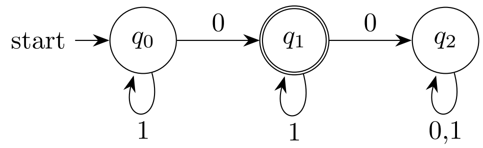
Solution

4. *Prove that $L = \{0^n10^n \mid n \in \mathbb{N}\}$ is not DFA-recognizable.*

Solution Suppose that this language is accepted by some deterministic finite automaton with N states. Consider the string $x = 0^n10^n$. Since x is in the language and $|x| \geq N$, by the Pumping Lemma, there exist strings u, v, and w such that $x = uvw$, $|v| \geq 1$, $|uv| \leq N$, and M accepts uv^iw for all $i > 0$. Since $|uv| \leq N$, it must be the case that uv is a string of 0's, and that w contains the 1 in 0^N10^N. Thus, if $i > 1$, uv^iw has more than N 0s to the left of the 1 and only N 0s to the right of the 1, and thus is not in the language. This contradicts the assumption that the language is accepted by M (since M accepts a string not in the language).

5. *Build a deterministic finite automaton that recognizes the set of strings of 0's and 1's, that only contain a single 0 (and any number of 1's). Describe the set of strings that lead to each state.*

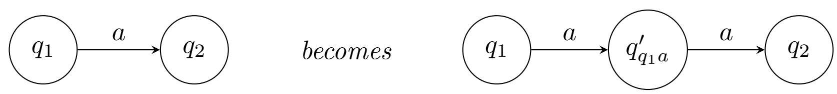
Solution



The strings leading to q_i contain i 0's.

6. *Given a string x, we can define the "character doubling" of x to be x with every character doubled: for example $cd(abc) = aabbcc$. Formally, $cd(\epsilon) = \epsilon$, and $cd(xa) = cd(x)aa$. We can then define the "character doubling" of a language L to be the set of all strings formed by doubling the characters of strings in L; formally $cd(L) = \{cd(x) \mid x \in L\}$.*

   Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, we can construct a new DFA M_{cd} that recognizes $cd(L(M))$ by adding a new state q'_{qa} to the middle of every transition from q on character a:



   (a) *Formally describe the components $(Q_{cd}, \Sigma_{cd}, \delta_{cd}, q_{0cd}, F_{cd})$ of M_{cd} in terms of the components of M. Be sure to describe δ_{cd} on all inputs (you may need to add one or more additional states).*

   Solution $Q_{cd} = Q \cup \{q'_{qa} \mid q \in Q, a \in A\} \cup \{X\}$

   $\delta_{cd} : (q, a) \mapsto q'_{qa}$; $\delta_{cd} : (q'_{qa}, a) \mapsto \delta(q, a)$; $\delta_{cd} : (q'_{qa}, b) \mapsto X$ if $a \neq b$, and $\delta_{cd} : (X, a) \mapsto X$.

   The remaining components are unchanged: $\Sigma_{cd} = \Sigma$, $q_{0cd} = q_0$, and $F_{cd} = F$.

   (b) *Use structural induction on x to prove that for all x, $\widehat{\delta}(q_0, x) = \widehat{\delta}_{cd}(q_{0cd}, cd(x))$.*

   Solution Let $P(x)$ be the statement that $\widehat{\delta}(q_0, x) = \widehat{\delta}_{cd}(q_0, cd(x))$. I will prove $\forall x, P(x)$ by structural induction.

   To show $P(\epsilon)$, note that $\widehat{\delta}(q_0, \epsilon) = q_0$. Moreover, $cd(\epsilon) = \epsilon$, so $\widehat{\delta}_{cd}(q_0, cd(\epsilon)) = \widehat{\delta}_{cd}(q_0, \epsilon) = q_0 = \widehat{\delta}(q_0, \epsilon)$, as required.

   To show $P(xa)$, we assume the inductive hypothesis $P(x)$. we compute:

$$
\begin{aligned}
\widehat{\delta}_{cd}(q_0, cd(xa)) &= \widehat{\delta}_{cd}(q_0, cd(x)aa) && \text{by definition of } cd \\
&= \delta_{cd}(\delta_{cd}(\widehat{\delta}_{cd}(q_0, cd(x)), a), a) && \text{by definition of } \widehat{\delta}_{cd} \\
&= \delta_{cd}(\delta_{cd}(\widehat{\delta}(q_0, x), a), a) && \text{by definition of } \widehat{\delta}_{cd} \\
&= \delta_{cd}(q_{(\widehat{\delta}(q_0, x))a}, a) && \text{by definition of } \delta_{cd} \\
&= \delta(\widehat{\delta}(q_0, x), a) && \text{by definition of } \delta_{cd} \\
&= \widehat{\delta}(q_0, xa) && \text{by definition of } \widehat{\delta}
\end{aligned}
$$

   (c) *We can also define the "string doubling" of x to be xx. For example, $sd(abc) = abcabc$. Show that the set of regular languages is* not *closed under string doubling. In other words, give a regular language L and prove that $sd(L) = \{sd(x) \mid x \in L\}$ is not regular.*

   You can use any theorem proved in class to help prove this result.

Solution Let $L = 0^*1$. Clearly L is regular. Moreover, $sd(L) = \{0^n10^n1 \mid n \in \mathbb{N}\}$.

This language is not regular. To see this, assume for the sake of contradiction that it is. Then there exists some natural number m as in the pumping lemma. Let $x = 0^m10^m1$. Clearly $x \in sd(L)$, and $|x| \geq m$, so we can split x into u, v, and w, as in the pumping lemma. We know that $|uv| \leq n$, so v can only contain 0's. Then $x' = uv^2w$ contains more 0's before the first 1 than after, and thus $x' \notin sd(L)$. But the pumping lemma says that $x' \in sd(L)$; this is a contradiction, and thus $sd(L)$ is not regular.

7. *Happy Cat has been shown the following proof, and has promptly turned into Grumpy Cat. Briefly but clearly identify the error which has induced grumpiness.*

   To prove: *The language of the regular expression* 0^*1^* *is, in fact, not DFA-recognizable.*

   Proof. *Let L be the language of* 0^*1^*. *Assume there is some DFA* M *with n states that recognizes* L. *Let* $x = 0^{n-1}11$. *Clearly,* $x \in L$ *and* $|x| \geq n$. *Therefore according to the Pumping Lemma, we can split* x *into three parts* u, v *and* w, *such that* $|uv| \leq n$, $|v| \geq 1$, *and* $uv^iw \in L$ *for all natural numbers* i. *Let* $|v| = n$. *Since* $|uv| \leq n$, *it must be the case that* $u = \epsilon$, *and* $v = 0^{n-1}1$. *Then* $uv^2w = 0^{n-1}10^{n-1}11$, *which is clearly not in* L. *This contradicts our assumption that there is a DFA which recognizes the language.*

Solution The pumping lemma says there exists some v, but we have chosen a specific v. It may be that the pumping lemma gives some other v (such as 0).