Discrete Structures CS2800	Final Exam

Instructions: This is a 150 minute test. Answer the following questions in the provided booklet. Ensure that your name and netid are on your exam booklet. You may answer the questions in any order, but please mark that questions clearly. Books, notes, calculators, laptops, and carrier pigeons are all disallowed. You may leave mathematical expressions unevaluated (e.g. just write $17 \cdot 3$ instead of 51 and don't bother evaluating $C(17, 3)$). Good luck!

1. (a) [3 points] How many functions are there from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$.

   (b) [3 points] How many of these functions are one-to-one?

   (c) [2 points] How many of these functions are onto?

2. [10 points] Prove by induction that $3^{2n-1} + 1 \equiv 0 \pmod 4$ for all $n \geq 1$.

3. (a) [1 points] What are the units of $\mathbb{Z}_{11}$?

   (b) [1 points] What is $\phi(11)$?

   (c) [4 points] Use Euler's theorem to compute $18^{1922} \bmod 11$.

4. (a) [3 points] In how many ways can three computer scientists and two geologists be chosen from a group of seven computer scientists and six geologists to visit an oil rig? (There's no need to simplify your answer.)

   (b) [3 points] What is the probability that Alice (a computer scientist) and Bob (a geologist) will both be chosen, if each possible combination of three computer scientists and two geologists is equally likely?

5. [4 points] There are 52 people at a party, all between the ages of 1 and 100. Use the pigeonhole principle to prove that there are either two people of the same age, or two people whose ages are consecutive integers.

6. Two fair dice are rolled. Suppose that you describe the outcomes using the sample space $\{(i, j) : 1 \leq i \leq 6, 1 \leq j \leq 6\}$, with each element being equally likely. Consider the following random variables:

$$X_1(i,j) = \begin{cases} 1 & \text{if } i = 6 \\ 0 & \text{otherwise,} \end{cases}$$
$$X_2(i,j) = \begin{cases} 1 & \text{if } j = 6 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) [1 points] What does X_1 represent (in English)? What does X_2 represent?

   (b) [1 points] Let $X = X_1 + X_2$. What does X represent (in English)?

   (c) [2 points] We claim that X_1 and X_2 are independent. What would you have to check to confirm that?

   (d) [3 points] What is $E(X)$?

7. [5 points] A simplified form of Bayes's rule is given by the following expression:

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

   Prove this identity.

8. (This is a variant of a problem due to Lewis Carroll, who wrote "Alice in Wonderland".) A bag has a white ball in it. A second ball is put into the bag, which is white with probability 2/3 and black with probability 1/3 (so with probability 2/3, there are two white balls in the bag and with probability 1/3 there is a white ball and a black ball). Now a ball is chosen from the bag at random.

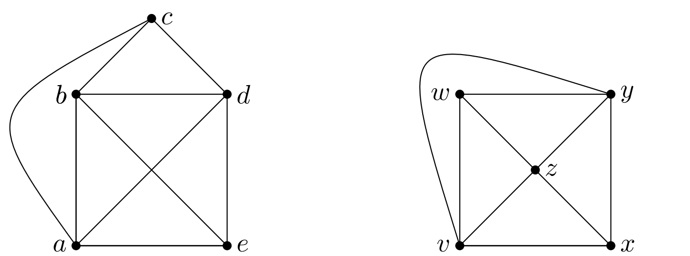
(a) [2 points] Carefully describe a sample space for this problem.

(b) [2 points] What is the probability of each element of this sample space.

(c) [3 points] If the ball chosen is white, what is the probability that the second ball is white?

9. [3 points] If an undirected graph has chromatic number 2, is it bipartite? Why or why not?

10. [3 points] Are the two graphs below isomorphic? If you think that they are isomorphic, give the bijection f between the graphs that demonstrates the isomorophism. If you think that they are not isomorphic, explain why.



11. [3 points] What is the clique number of the graph above on the left? (Explain your answer.)

12. [6 points] Define the relation $\sim$ on $N \times N$ by taking $(m, n) \sim (k, l)$ if $m + l = n + k$. Show that $\sim$ is an equivalence relation.

13. [5 points] Build a deterministic finite automaton that recognizes the set of strings of 0's and 1's, that only contain a single 0 (and any number of 1's). Describe the set of strings that lead to each state.

14. [10 points] Given a string x, we can define the "character doubling" of x to be x with every character doubled: for example $cd(abc) = aabbcc$. Formally, $cd(\epsilon) = \epsilon$, and $cd(xa) = cd(x)aa$. We can then define the "character doubling" of a language L to be the set of all strings formed by doubling the characters of strings in L; formally $cd(L) = \{cd(x) \mid x \in L\}$.

    Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, we can construct a new DFA M_{cd} that recognizes $cd(L(M))$ by adding a new state q'_{qa} to the middle of every transition from q on character a:

    $$q_1 \xrightarrow{a} q_2 \qquad \text{becomes} \qquad q_1 \xrightarrow{a} q'_{q_1 a} \xrightarrow{a} q_2$$

    (a) Formally describe the components $(Q_{cd}, \Sigma_{cd}, \delta_{cd}, q_{0cd}, F_{cd})$ of M_{cd} in terms of the components of M. Be sure to describe δ_{cd} on all inputs (you may need to add one or more additional states).

    (b) Use structural induction on x to prove that for all x, $\widehat{\delta}(q_0, x) = \widehat{\delta}_{cd}(q_{0cd}, cd(x))$.

15. [10 points] We can also define the "string doubling" of x to be xx. For example, $sd(abc) = abcabc$. Show that the set of regular languages is *not* closed under string doubling. In other words, give a regular language L and prove that $sd(L) = \{sd(x) \mid x \in L\}$ is not regular.

    You can use any theorem proved in class to help prove this result.

16. (a) [6 points] Build a proof tree showing that $\vdash \neg(P \vee Q) \rightarrow \neg P$. You may refer to the list of rules given at the end of the exam. (Hint:

    (b) [5 points] Show, using truth tables, that $\models \neg(P \vee Q) \rightarrow \neg P$.

17. [4 points] Translate the following expressions into first-order logic, making clear what the predicates you use stand for.

    (a) Everyone is on someone's contact list.

    (b) No one is on everyone's contact list.

18. [4 points] Which of the following formulas is true if the domain is the natural numbers, and which are true if the domain is the real numbers. (Explain your answer in each case.)

   (a) $\exists x \exists y (x < y \wedge \forall z (z \leq x \vee y \leq z))$

   (b) $\exists x \exists y (2x - y = 4 \wedge 2x + y = 6)$.

Appendix: natural deduction proof rules

$$\frac{}{\cdots \vdash \varphi \vee \neg\varphi} \text{ (excl mid)} \qquad \frac{}{\cdots, \varphi \vdash \varphi} \text{ (assum)}$$

$$\frac{\cdots \vdash \varphi \quad \cdots \vdash \neg\varphi}{\cdots \vdash \psi} \text{ (absurd)}$$

$$\frac{\cdots \vdash \varphi \wedge \psi}{\cdots \vdash \varphi} \text{ ($\wedge$ elim)} \qquad \frac{\cdots \vdash \varphi \wedge \psi}{\cdots \vdash \psi} \text{ ($\wedge$ elim)} \qquad \frac{\cdots \vdash \varphi \quad \cdots \vdash \psi}{\cdots \vdash \varphi \wedge \psi} \text{ ($\wedge$ intro)}$$

$$\frac{\cdots \vdash \varphi_1 \vee \varphi_2 \quad \cdots, \varphi_1 \vdash \psi \quad \cdots, \varphi_2 \vdash \psi}{\cdots \vdash \psi} \text{ ($\vee$ elim)}$$

$$\frac{\cdots \vdash \varphi}{\cdots \vdash \varphi \vee \psi} \text{ ($\vee$ intro)} \qquad \frac{\cdots \vdash \psi}{\cdots \vdash \varphi \vee \psi} \text{ ($\vee$ intro)}$$

$$\frac{\cdots \vdash \varphi \quad \cdots \vdash \varphi \to \psi}{\cdots \vdash \psi} \text{ ($\to$ elim)} \qquad \frac{\cdots, \varphi \vdash \psi}{\cdots \vdash \varphi \to \psi} \text{ ($\to$ intro)}$$