Number Representations and the Division Algorithm

CS 2800: Discrete Structures, Spring 2015

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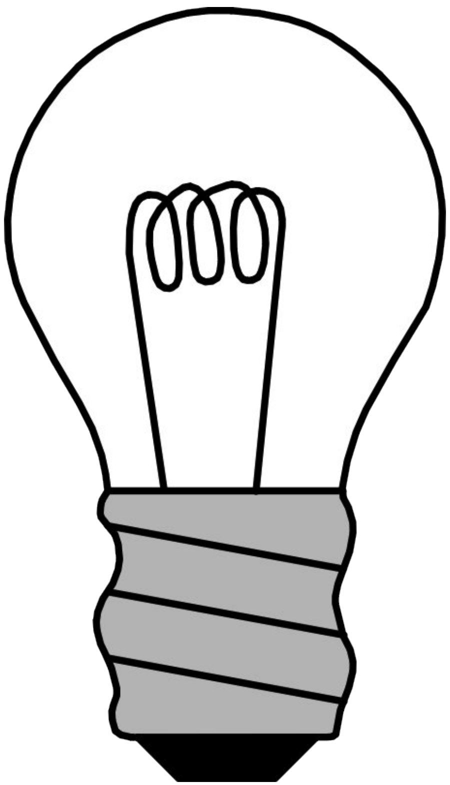
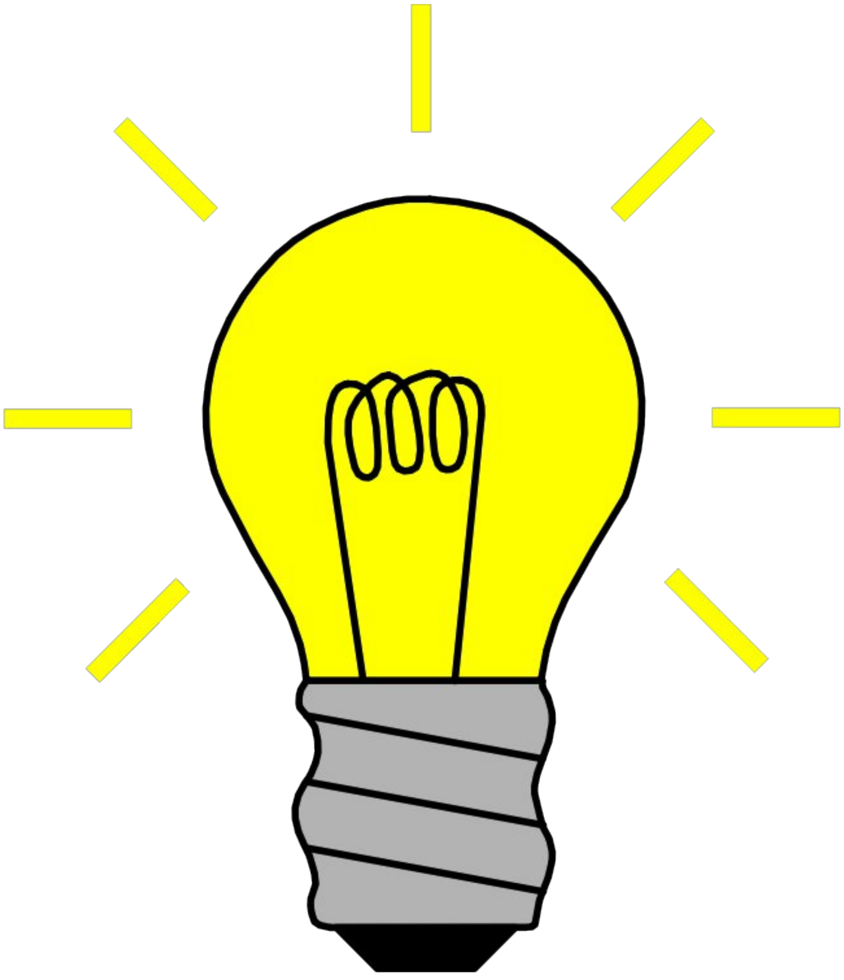
There are only 10 types of people in the world.

Those who understand binary, and those who don't

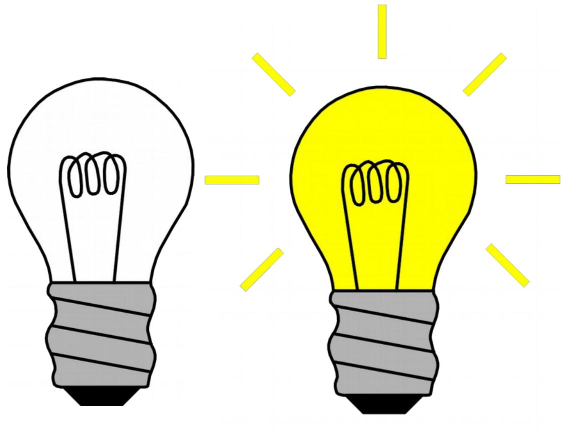
Binary Digits

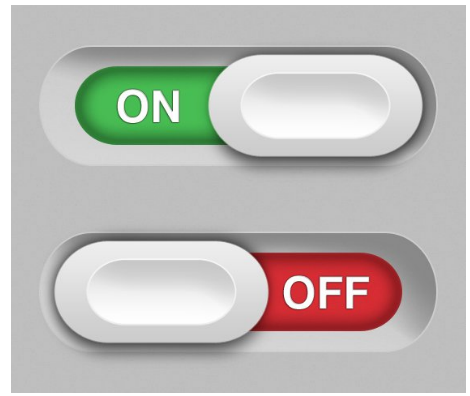


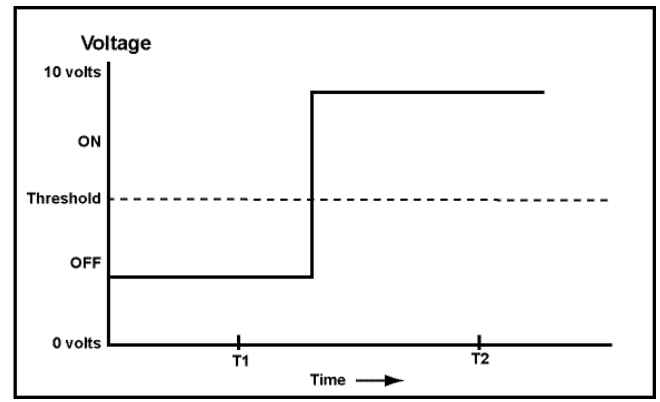
Binary Digits



Binary Digits







True
False

0
1

?

openclipart.org, imgbuddy.com, chortle.ccsu.edu, speechdudes.wordpress.com, ncetm.org.uk

Binary representations of numbers

$$1\ 0\ 1\ 0\ 1_2$$

$$1\times2^4 + 0\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0$$

$$= 21_{10}$$

Numbers in base b

Each a_i is a digit between 0 and $b-1$

$$\overbrace{a_4 \; a_3 \; a_2 \; a_1 \; a_0}$$

$$a_4 \times b^4 + a_3 \times b^3 + a_2 \times b^2 + a_1 \times b^1 + a_0 \times b^0$$

Common bases:

Binary (2), **Ternary** (3), **Octal** (8), **Decimal** (10), **Hexadecimal** (16)

All rules of arithmetic remain exactly the same,
just remember 10_b is b

Common bases

- **Binary** (base 2)
  - Digits: 0, 1
- **Ternary** (base 3)
  - Digits: 0, 1, 2
- **Octal** (base 8)
  - Digits: 0, 1, 2, 3, 4, 5, 6, 7
- **Decimal** (base 10)
  - Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- **Hexadecimal** (base 16)
  - Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A (= 10_{10}), B (=11_{10}), C (=12_{10}), D (=13_{10}), E (=14_{10}), F (=15_{10})

Conversions to/from decimal

- Converting from base b to decimal
  - Add up the powers of b as in the previous slide
- Converting from decimal to base b
  - Divide by b and write down the remainder
  - Repeat with the quotient, writing down the remainders ***right*** to ***left***

Example conversions worked out on the board

“Division Algorithm” (not really an algorithm)

- Theorem: Given any integer a, and a positive integer b, there exist integers q (the “quotient”), and r (the “remainder”), such that
  - $0 \leq r < b$, and
  - $a = qb + r$

- Proof: By induction!

we'll prove it only for non-negative a –
the proof for negative a is similar

Proof of Division Algorithm

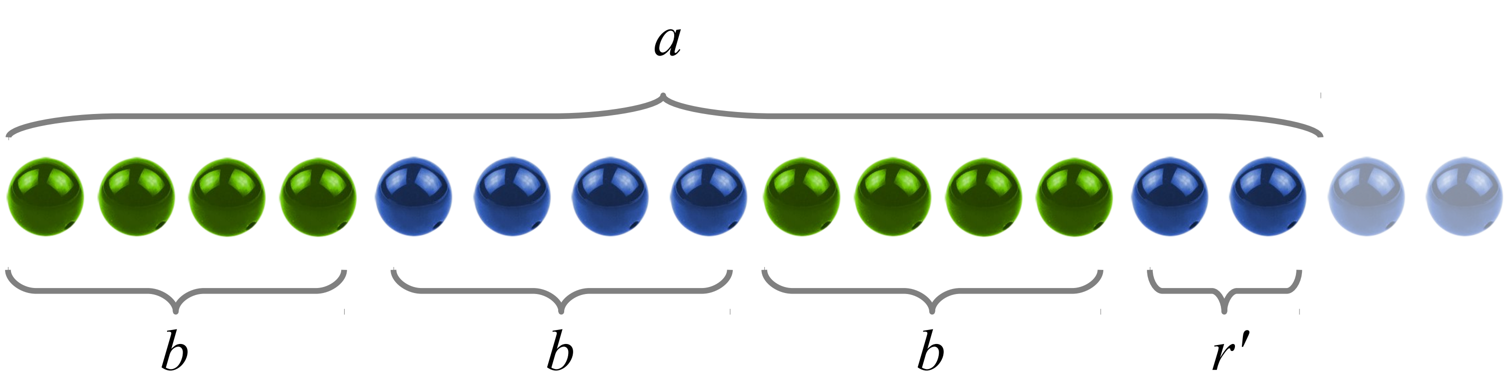
(for non-negative a)

- We will do induction on a
- $S(a)$ = "for the given a, and any b, the theorem is true"
- Base case:
  - When $a = 0$, choose $q = 0$, $r = 0$
  - Clearly $0 \leq r < b$ (since $b > 0$) and $a = qb + r$
- Inductive hypothesis: Given a, we have $a = q'b + r'$ for q' and r' satisfying the conditions

Proof of Division Algorithm

(for non-negative a)

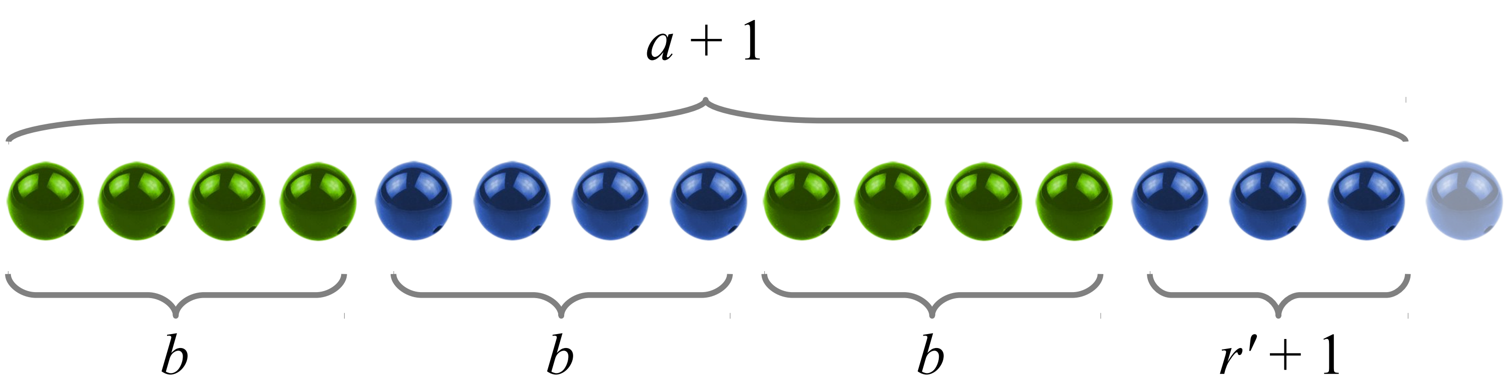
- Inductive step: Two cases
  - Case 1: $r' < b - 1$



Proof of Division Algorithm

(for non-negative a)

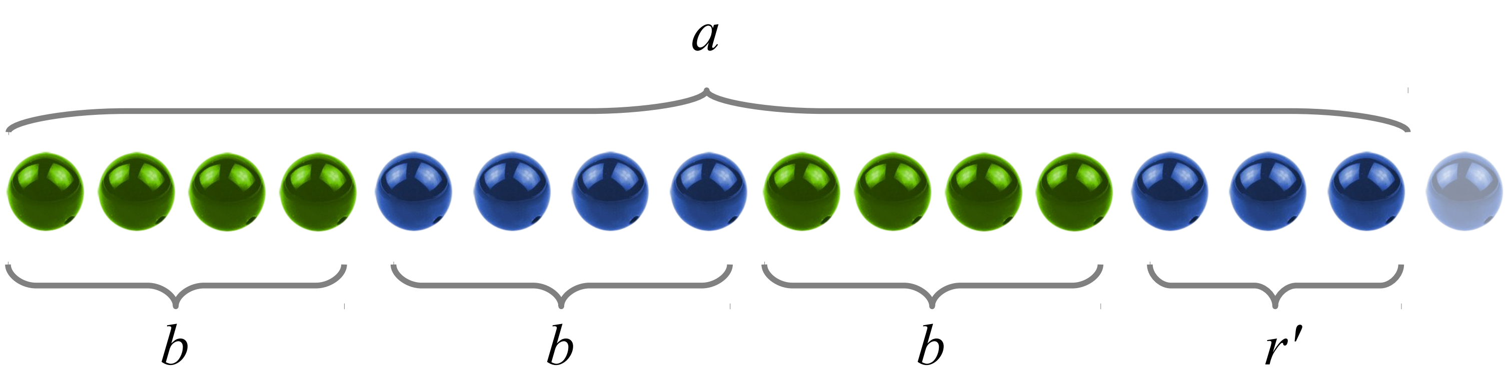
- Inductive step: Two cases
  - Case 1: $r' < b - 1$
    - Choose $q = q', r = r' + 1$
    - Clearly $0 \leq r < b$
    - ... and $a + 1 = q'b + r' + 1 = q'b + (r' + 1) = qb + r$



Proof of Division Algorithm

(for non-negative a)

- Inductive step: Two cases
  - Case 2: $r' = b - 1$

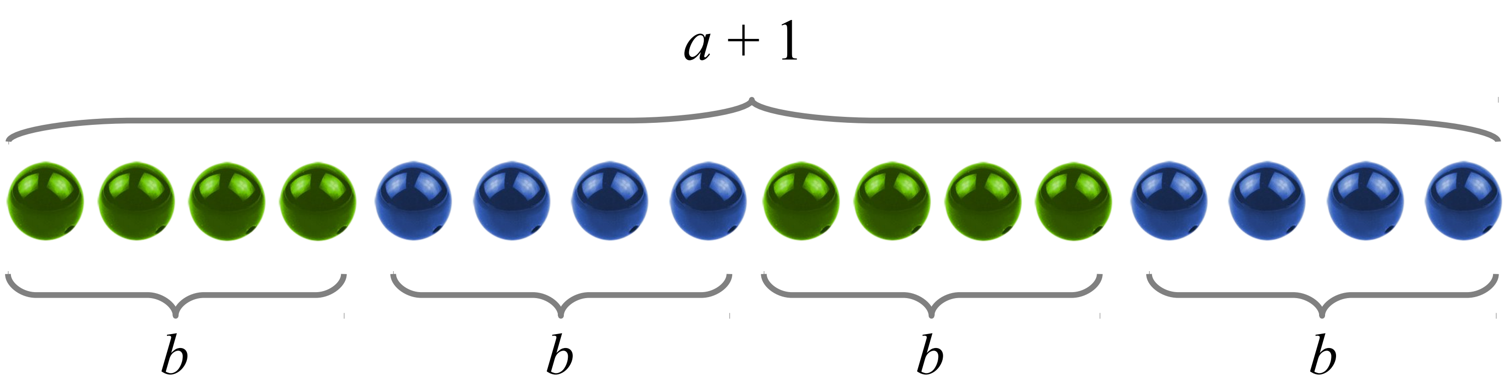


Proof of Division Algorithm

(for non-negative a)

- Inductive step: Two cases
  - Case 2: $r' = b - 1$
    - Choose $q = q' + 1, r = 0$
    - Clearly $0 \leq r < b$
    - … and $a + 1 = q'b + r' + 1 = q'b + (b - 1) + 1 = (q' + 1)b + 0$
      $= qb + r$

Hence proved by induction!



Thought for the Day #1

Write out the proof for negative a

Quotient and Remainder are Unique

- Proof: Assume $a = qb + r = q'b + r'$
  - Then $(q - q')b = r' - r$
  - Since r and r' are between 0 and $b - 1$, we have

    $-b < (r' - r) < b$
  - Hence $-b < (q - q')b < b$
  - Since $b > 0$, we can divide to get

    $-1 < (q - q') < 1$
  - Hence $q = q'$ (since $q - q'$ is an integer)
  - ... and $r = r'$ (since $r' - r = (q - q')b = 0$)
- Implies that the representation of a number in a given base is also unique! (Prove!)