Bijections and Cardinality

CS 2800: Discrete Structures, Spring 2015

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Recap: Left and Right Inverses

- A function is ***injective*** (one-to-one) iff it has a ***left inverse***
  - $g: B \to A$ is a left inverse of $f: A \to B$ if $g(f(a)) = a$ for all $a \in A$

- A function is ***surjective*** (onto) iff it has a ***right inverse***
  - $h: B \to A$ is a right inverse of $f: A \to B$ if $f(h(b)) = b$ for all $b \in B$

Thought for the Day #1

Is a left inverse injective or surjective? Why?

Is a right inverse injective or surjective? Why?

(Hint: how is f related to its left/right inverse?)

Sur/injectivity of left/right inverses

- The left inverse is always surjective!
  - … since f is its right inverse
- The right inverse is always injective!
  - … since f is its left inverse

Factoid for the Day #1

If a function has both a left inverse and a right inverse, then the two inverses are identical, and this common inverse is unique

(Prove!)

This is called the *two-sided inverse*, or usually just the **inverse** f^{-1} of the function f

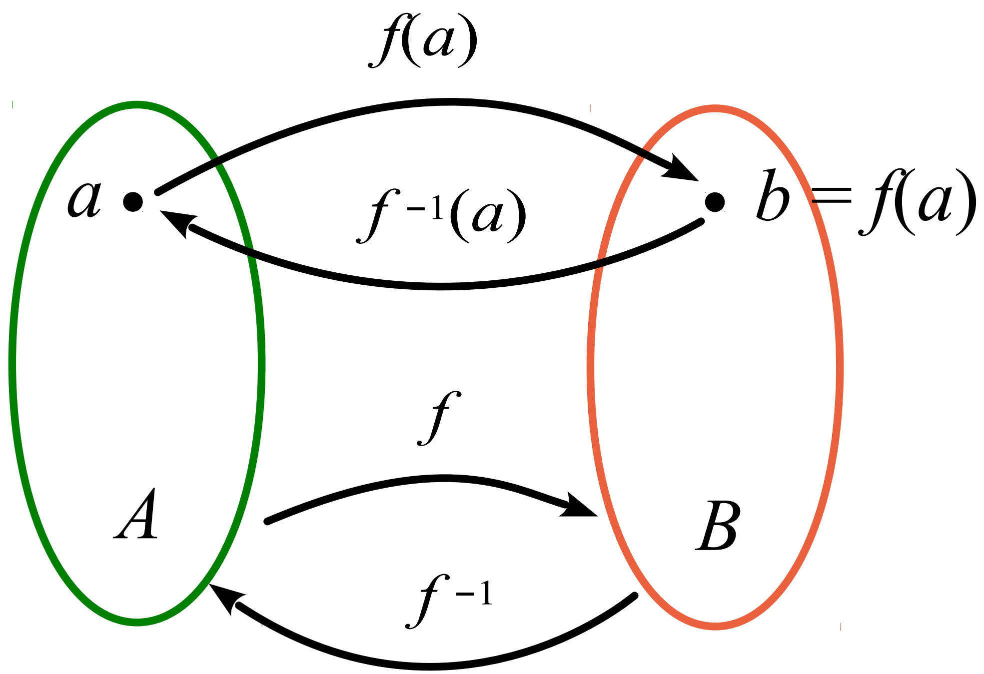
http://www.cs.cornell.edu/courses/cs2800/2015sp/handouts/jonpak_function_notes.pdf

Bijection and two-sided inverse

- A function f is bijective iff it has a two-sided inverse
- **Proof** ($\Longrightarrow$): If it is bijective, it has a left inverse (since injective) and a right inverse (since surjective), which must be one and the same by the previous factoid
- **Proof** ($\Longleftarrow$): If it has a two-sided inverse, it is both injective (since there is a left inverse) and surjective (since there is a right inverse). Hence it is bijective.

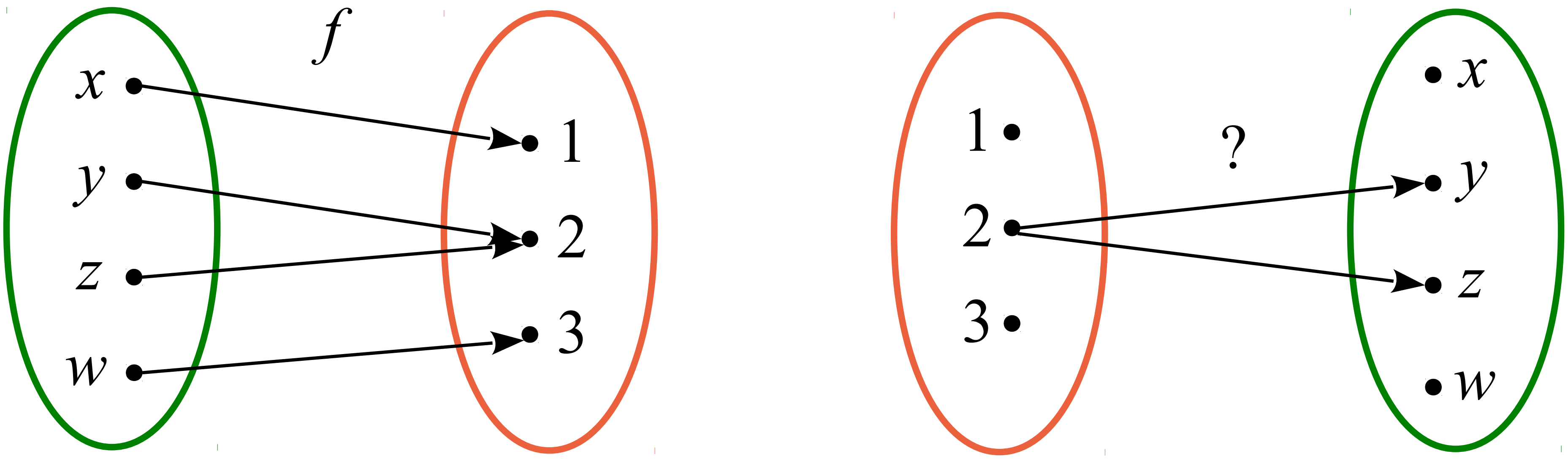
Inverse of a function

- The inverse of a bijective function $f : A \longrightarrow B$ is the *unique* function $f^{-1}: B \longrightarrow A$ such that for any $a \in A$, $f^{-1}(f(a)) = a$ and for any $b \in B$, $f(f^{-1}(b)) = b$
- A function is bijective iff it has an inverse function



Inverse of a function

- If f is not a bijection, it cannot have an inverse function

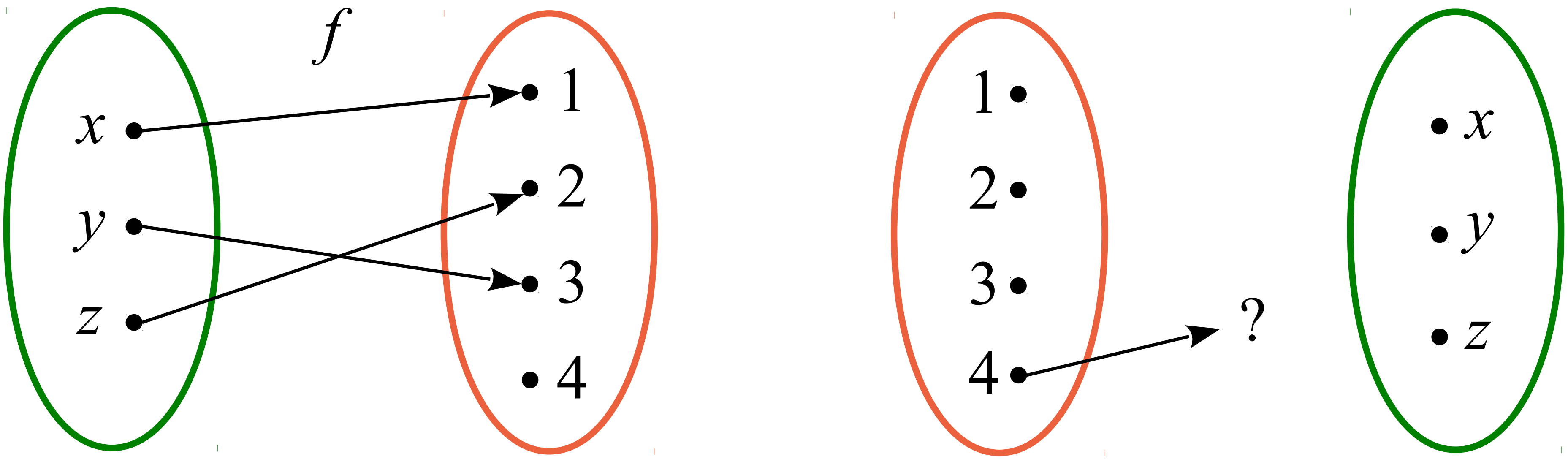


Onto, not one-to-one

$f^{-1}(2) = ?$

Inverse of a function

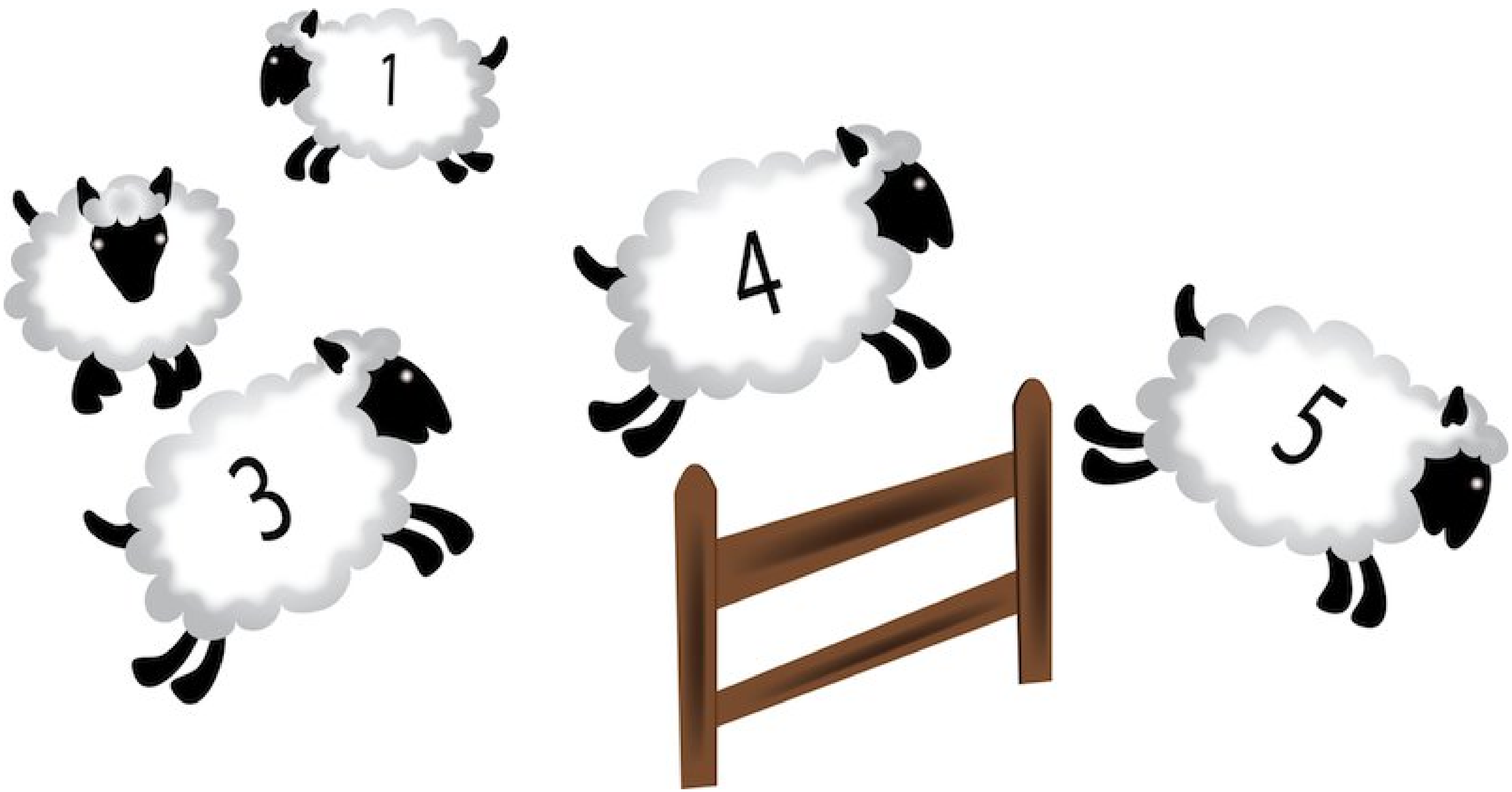
- If f is not a bijection, it cannot have an inverse function



One-to-one, not onto

$f^{-1}(4) = ?$

How can we count elements in a set?



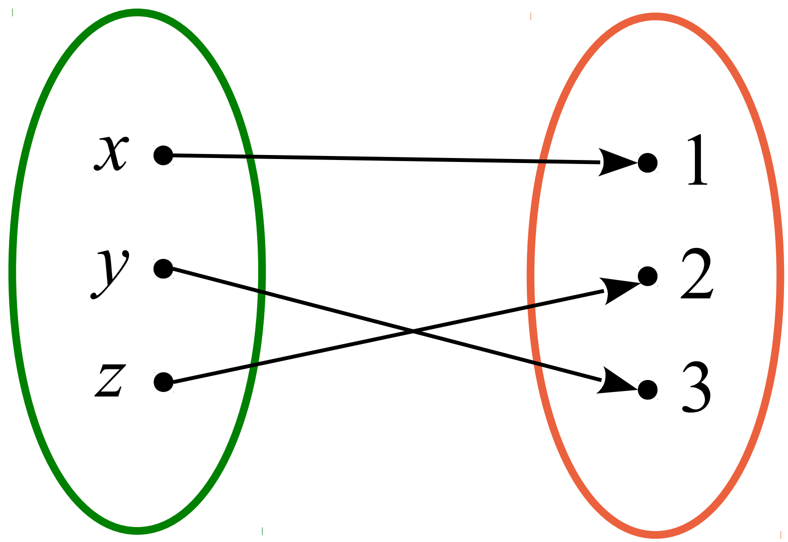
How can we count elements in a set?

- Easy for finite sets – just count the elements!
- Does it even make sense to ask about the number of elements in an infinite set?
- Is it meaningful to say one infinite set is larger than another?
  - Are the natural numbers larger than
    - the even numbers?
    - the rational numbers?
    - the real numbers?

Cardinality and Bijections

- If A and B are finite sets, clearly they have the same number of elements iff there is a bijection between them

e.g. $|\{x, y, z\}| = |\{1, 2, 3\}| = 3$

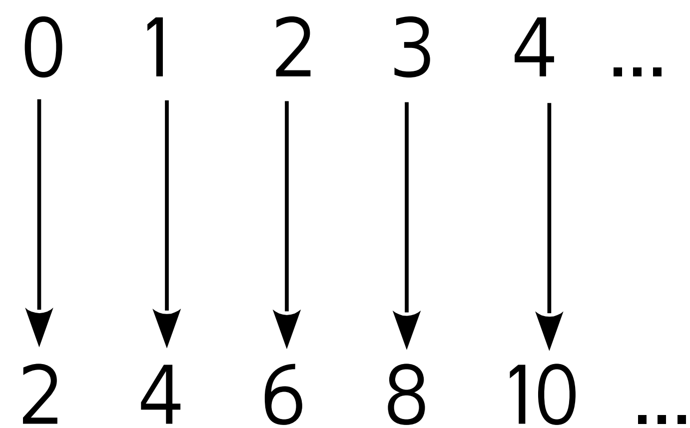


Cardinality and Bijections

- **Definition:** Set A has the same ***cardinality*** as set B, denoted $|A| = |B|$, iff there is a bijection from A to B
  - For finite sets, cardinality is the number of elements
  - There is a bijection from n-element set A to $\{1, 2, 3, \ldots, n\}$

Cardinality and Bijections

- Natural numbers and even numbers have the same cardinality

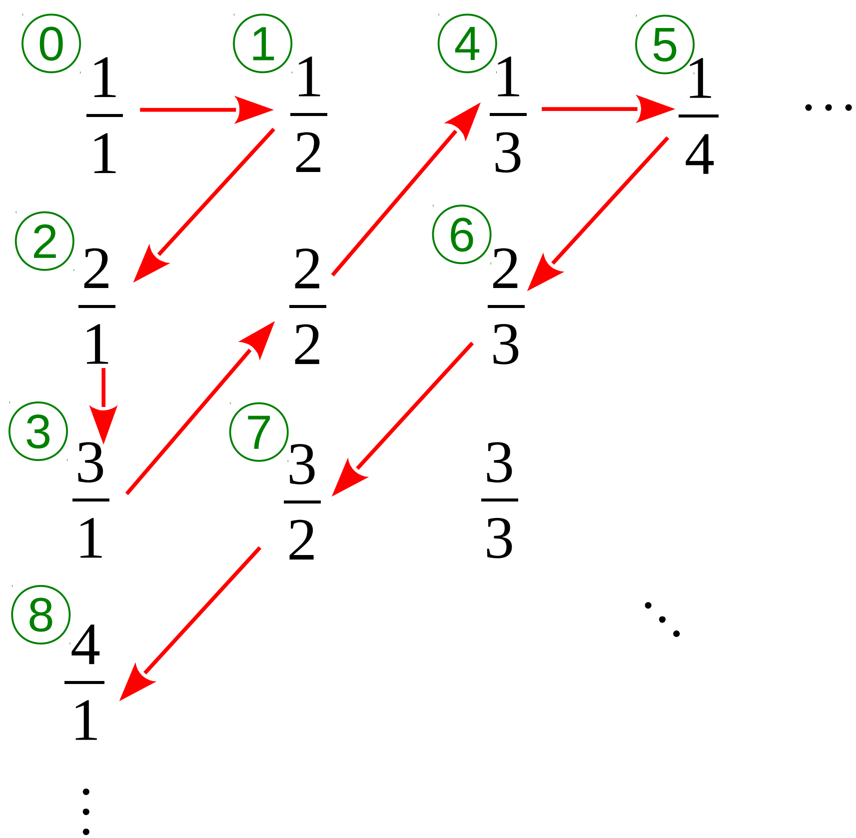


Sets having the same cardinality as the natural numbers (or some subset of the natural numbers) are called <u>countable sets</u>

Cardinality and Bijections

- Natural numbers and rational numbers have the same cardinality!

Illustrating proof only for positive rationals here, can be easily extended to all rationals



Cardinality and Bijections

- The natural numbers and real numbers *do not* have the same cardinality

x_1	$0\,.\,0\,0\,0\,0\,0\,0\,0\,0\,0\,\ldots$
x_2	$0\,.\,1\,0\,3\,0\,4\,0\,5\,0\,1\,\ldots$
x_3	$0\,.\,9\,8\,7\,6\,5\,4\,3\,2\,1\,\ldots$
x_4	$0\,.\,0\,1\,2\,1\,2\,1\,2\,1\,2\,\ldots$
x_5	$\vdots$

Cardinality and Bijections

- The natural numbers and real numbers *do not* have the same cardinality

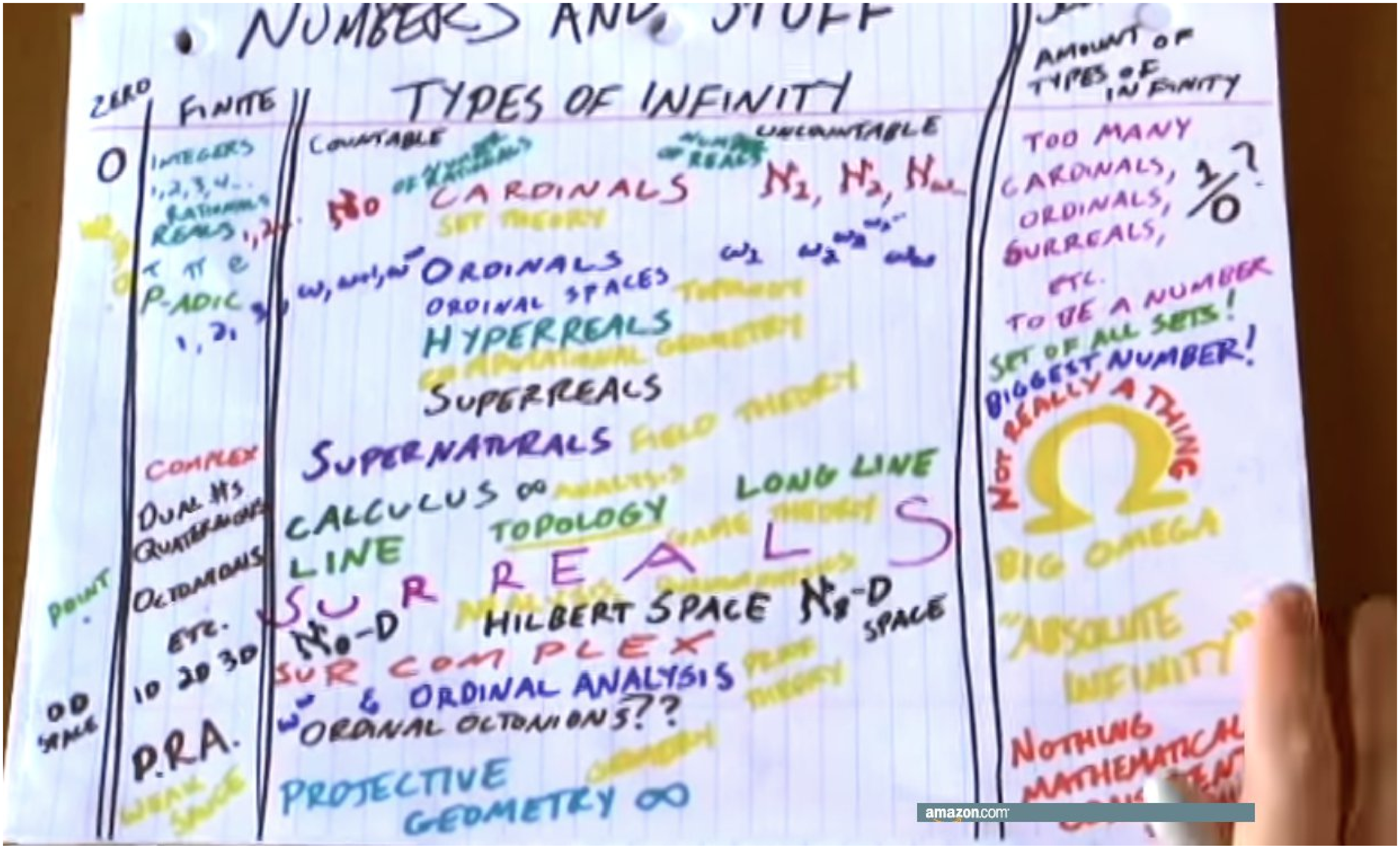
x_1	0 . **0** 0 0 0 0 0 0 0 0 …
x_2	0 . 1 **0** 3 0 4 0 5 0 1 …
x_3	0 . 9 8 **7** 6 5 4 3 2 1 …
x_4	0 . 0 1 2 **1** 2 1 2 1 2 …
x_5	⋮

Consider the number
$y = 0\,.\,b_1\,b_2\,b_3\ldots$

$$b_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ decimal place of } x_i \text{ is zero} \\ 0 & \text{if it is non-zero} \end{cases}$$

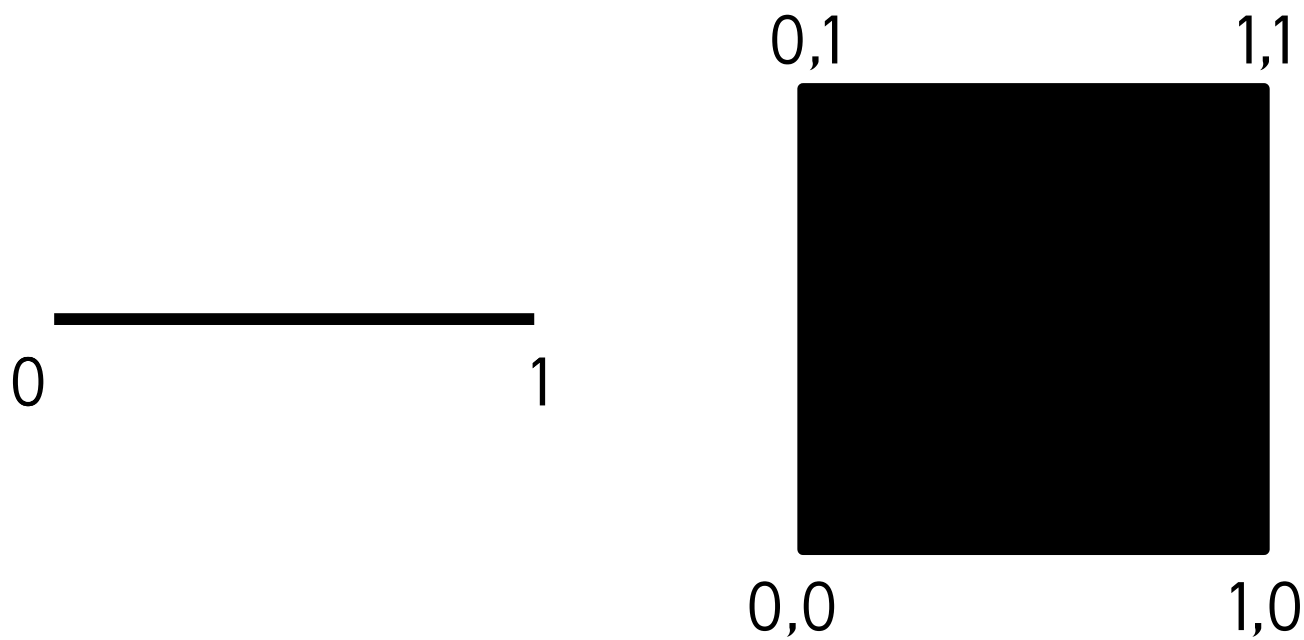
y cannot be equal to any x_i – it differs by one digit from each one!

There are many infinities



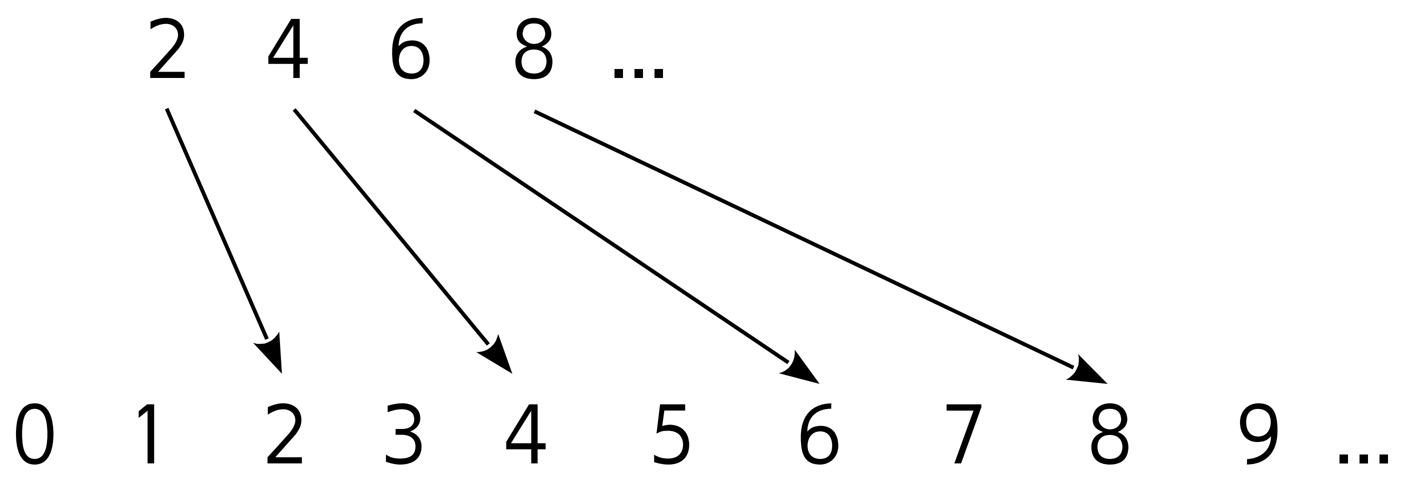
Thought for the Day #2

Do the real interval $[0, 1]$ and the unit square $[0, 1] \times [0, 1]$ have the same cardinality?



Comparing Cardinalities

- **Definition:** If there is an injective function from set A to set B, we say $|A| \leq |B|$



$$|Evens| \leq |\mathbb{N}|$$

Comparing Cardinalities

- **Definition:** If there is an injective function from set A to set B, but not from B to A, we say $|A| < |B|$
- Cantor–Schröder–Bernstein theorem: If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$
  - Exercise: prove this!
  - i.e. show that there is a bijection from A to B iff there are injective functions from A to B and from B to A
  - (it's not easy!)