Discrete Structures	Homework 5
CS2800 Spring 2015	Functions

1. Each of the following relations has domain $\{x, y, z\}$ and codomain $\{1, 2, 3\}$. For each, state whether it is a function or not. If it is a function, state (i) its image, (ii) whether it is surjective or not, (iii) whether it is injective or not. If it is not a function, state why not.
   (a) $\{(x, 1), (y, 2), (x, 3), (z, 2)\}$
   (b) $\{(x, 1), (y, 2)\}$
   (c) $\{(x, 3), (y, 2), (z, 1)\}$
   (d) $\{(x, 1), (y, 1), (z, 1)\}$
2. Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is injective (one-to-one) or not. If it is not injective, give a counterexample. Note that $\mathbb{Z}$ is the set of all integers: negative, positive or zero.
   (a) $f(n) = n - 1$
   (b) $f(n) = n^2 + 1$
   (c) $f(n) = n^3$
   (d) $f(n) = \lceil n/2 \rceil$ ($\lceil x \rceil$ denotes the smallest integer greater than or equal to x)
3. Recall that $[X \to Y]$ denotes the set of all functions with domain X and codomain Y.
   (a) Give a bijection from $[X \to Y] \times [X \to Z]$ to $[X \to Y \times Z]$.
   (b) Give a bijection from $[X \to [Y \to Z]]$ to $[X \times Y \to Z]$.
   (c) Assuming there is a bijection between X and Y, give a bijection from $[X \to Z]$ to $[Y \to Z]$.
4. (a) A function $f : A \to A$ is called involutive if for all $x \in A$, $f(f(x)) = x$. Prove or disprove:
      i. if f is involutive, then it is injective.
      ii. if f is involutive, then it is surjective.
   (b) A function $f : A \to A$ is called idempotent if for all $x \in A$, $f(f(x)) = f(x)$. Prove or disprove:
      i. if f is idempotent, then it is injective.
      ii. if f is idempotent, then it is surjective.
   (c) If $f : B \to C$ and $g : A \to B$ are functions, then $f \circ g$ is the function from A to C defined by: $(f \circ g) : x \mapsto f(g(x))$. Prove or disprove: if f and $f \circ g$ are one-to-one, then g is one-to-one.
5. In the first few homeworks, we asserted various facts about sets. We will now prove two of them. Given two sets $A \subseteq S$ and $B \subseteq S$, show that
   (a) $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$.
   (b) $(A \setminus B) \cap (A \cap B) = \emptyset$.

   Note that the formal definition of equality for sets is that $A = B$ if $A \subseteq B$ and $B \subseteq A$, and the formal definition of subset is $A \subseteq B$ if for all $x \in A$, $x \in B$. Definitions for $\cup$, $\cap$, $\setminus$ and $\emptyset$ are on the lecture slides.