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| Discrete Structures | Prelim 2 |
| CS2800 | April 13, 2015 |

**Instructions:** This is a 50 minute exam. Please answer the following questions in the provided booklet. Ensure that your name and netid are on your exam booklet. Clearly indicate your answer to each question. Books, notes, calculators, laptops, and carrier pigeons are all disallowed. You may leave mathematical expressions unevaluated (e.g. just write $17 \cdot 3$ instead of 51).

**This exam has 5 (FIVE) QUESTIONS, totaling 13 POINTS, and an ADDITIONAL SECTION with useful definitions. Please turn the page over!**

1. **(3 points)** The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers $n$ (including 0):

$$\sum_{i=0}^{n} F_i \;=\; F_{n+2} - 1$$

2. **(3 points)** Prove that the following language, consisting of all strings of 1's whose length is a power of 2, is not DFA-recognizable:

$$L = \{1^n \mid n = 2^k \text{ for some natural number } k\}$$

You may use without proof the facts that $2^k > k$ and $2^k + k < 2^{k+1}$ for all natural numbers $k$.

(**Note:** Here, $1^n$ represents the string of $n$ 1's. But $2^k$ represents the $k^{\text{th}}$ power of 2, that is, the product of $k$ 2's.)

3. **(3 points)** Given DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$, we can construct a machine $M_{12}$ with $L(M_{12}) = L(M_1) \cap L(M_2)$ as follows:

   - Let $Q = Q_1 \times Q_2 =$ the set of all ordered pairs $(q_1, q_2)$, where $q_1 \in Q_1$ and $q_2 \in Q_2$.
   - Let $q_0 \in Q = (q_{01}, q_{02})$.
   - Let $F = F_1 \times F_2 = \{(q_1, q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2\}$.
   - Let $\delta_{12}((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.
   - Let $M_{12} = (Q, \Sigma, \delta_{12}, q_0, F)$.

Use structural induction to prove that for all $x \in \Sigma^*$, $\widehat{\delta}_{12}((q_1, q_2), x) = \left(\widehat{\delta}_1(q_1, x), \widehat{\delta}_2(q_2, x)\right)$.

4. **(4 points)**

   (a) **(2 points)** Draw a finite automaton (DFA, NFA or $\epsilon$-NFA) with alphabet $\{0, 1\}$ to recognize the language

   $$\{x \in \{0,1\}^* \mid x \text{ contains the substring } 010\}$$

   (b) **(2 points)** Draw a finite automaton (DFA, NFA or $\epsilon$-NFA) with alphabet $\{a, b\}$ to recognize the same language as the regular expression $(ab|ba)^*$.

5. **(0 points)** Prove or disprove: it is better for a cat to be grumpy than happy.

If you prove the statement, Sid will provide cookies after the final. If you disprove it, Mike will provide cookies. **Note:** It is inhumane to apply the Pumping Lemma to cats.

## Definitions

- Given a finite set $\Sigma$, the set $\Sigma^*$ is defined inductively by the following rules:
  1. $\epsilon \in \Sigma^*$.
  2. for all $x \in \Sigma^*$ and $a \in \Sigma$, $xa \in \Sigma^*$
- The set $RE$ of regular expressions with alphabet $\Sigma$ is defined inductively by the following rules:
  1. $\emptyset \in RE$
  2. $\epsilon \in RE$
  3. for all $a \in \Sigma$, $a \in RE$
  4. for all $r_1$, $r_2 \in RE$, the concatenation $r_1 r_2 \in RE$
  5. for all $r_1$, $r_2 \in RE$, the alternation $r_1 | r_2 \in RE$
  6. for all $r \in RE$, $r^*$ is also in $RE$.
- Given a transition function $\delta : Q \times \Sigma \to Q$, the extended transition function $\widehat{\delta} : Q \times \Sigma^* \to Q$ is defined inductively by
  1. $\widehat{\delta}(q, \epsilon) = q$
  2. $\widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$
- The *language* $L(M)$ of a DFA $M$ is the set of all strings that are accepted by $M$, or in other words, the set

$$L(M) = \{x \in \Sigma^* \mid \widehat{\delta}(q_0, x) \in F\}$$