Discrete Structures	Prelim 2
CS2800 Fall 2014	

Instructions: This is a 50 minute exam. Please answer the following questions in the provided booklet. Ensure that your name and netid are on your exam booklet. Clearly indicate your answer to each question.

Books, notes, calculators, laptops, and carrier pigeons are all disallowed. You may leave mathematical expressions unevaluated (e.g. just write $17 \cdot 3$ instead of 51). You may use results proved in class (or in the homeworks, or the previous prelim) without proof.

1. Prove or give a counterexample: if $L_1 \cap L_2$ is regular, then at least one of L_1 and L_2 must be regular.

2. (a) Draw a DFA with alphabet $\{0, 1\}$ that recognizes the language

$$\{x \in \{0,1\}^* \mid x \text{ ends in 11 and } |x| \text{ is a multiple of 3}\}$$

   Here $|x|$ denotes the number of characters in x.

   (b) Write a regular expression that recognizes the same language. Please use only the standard syntax introduced in class, avoid constructions like "0^5" or ".∗" or "1+".

3. Let R be an equivalence relation on a set A. Recall the definition of the equivalence class of x: $[x] = \{y \in A \mid x \; R \; y\}$.

   (a) Prove that for all $x \in A$, there exists $y \in A$ such that $x \in [y]$.

   (b) Prove that if $[x] \cap [y]$ is non-empty then $[x] = [y]$.

4. Recall that the Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined inductively as follows:

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 3 \end{aligned}$$

   That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove that:

$$\sum_{i=1}^{n} F_i^2 \; = \; F_n F_{n+1}$$

5. A chocolate bar consists of n identical square pieces arranged in an unbroken rectangular grid. For instance, a 12-piece bar might be a 3×4, 2×6 or 1×12 grid. A single snap breaks the bar along a straight line separating the squares, into two smaller rectangular pieces. Prove that regardless of the initial dimensions of the bar, any n-piece bar requires exactly $n - 1$ snaps to break it up into individual squares.

6. Let L_π be the language consisting of prefixes of the decimal expansion of π:

$$L_\pi = \{\text{‘3’}, \text{‘31’}, \text{‘314’}, \text{‘3141’}, \text{‘31415’}, \text{‘314159’}, \ldots\}$$

   Prove that L_π is not regular. You may use the fact that π cannot be written as a repeating decimal, that is, there are no sequences of digits $d_1, d_2, \ldots d_m$ and $e_1, e_2, \ldots e_n$ such that

$$\pi = d_1.d_2 \cdots d_n\overline{e_1 e_2 \cdots e_n} = \text{“}d_1.d_2 \cdots d_m(e_1 \cdots e_n)(e_1 \cdots e_n)(e_1 \cdots e_n) \cdots\text{”}$$