Fermat's Little Theorem

CS 2800: Discrete Structures, Fall 2014

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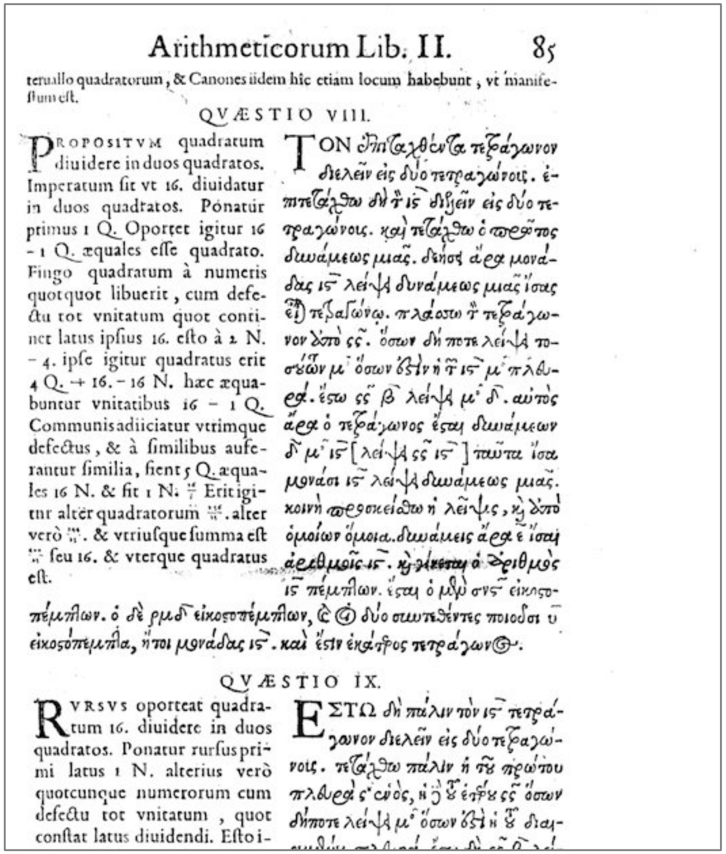
Not to be confused with...



Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 - 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. - 4. ipse igitur quadratus erit 4 Q. + 16. - 16 N. hæc æquabuntur vnitatibus 16 - 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. 16/5. Erit igitur alter quadratorum 256/25. alter verò 144/25. & vtriusque summa est 400/25 seu 16. & vterque quadratus est.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto i-

Fermat's <u>Last</u> Theorem:

$x^n + y^n = z^n$ has no integer solution for $n > 2$

Recap: Modular Arithmetic

- Definition: $a \equiv b \pmod{m}$ if and only if $m \mid a - b$
- Consequences:
  - $a \equiv b \pmod{m}$ iff $a \bmod m = b \bmod m$

    (congruence $\Leftrightarrow$ Same remainder)

  - If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
    - $a + c \equiv b + d \quad \pmod{m}$
    - $ac \equiv bd \quad \pmod{m}$

    (congruences can sometimes be treated like equations)

Fermat's Little Theorem

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$$a^p \equiv a \pmod{p}$$

Fermat's Little Theorem

- If p is a prime number, and a is any integer, then

$$a^p \equiv a \pmod{p}$$

- If a is not divisible by p, then

$$a^{p-1} \equiv 1 \pmod{p}$$

Fermat's Little Theorem

- Examples:
  - $21^7 \equiv 21 \pmod 7$

    ... but $21^6 \not\equiv 1 \pmod 7$

  - $111^{12} \equiv 1 \pmod{13}$

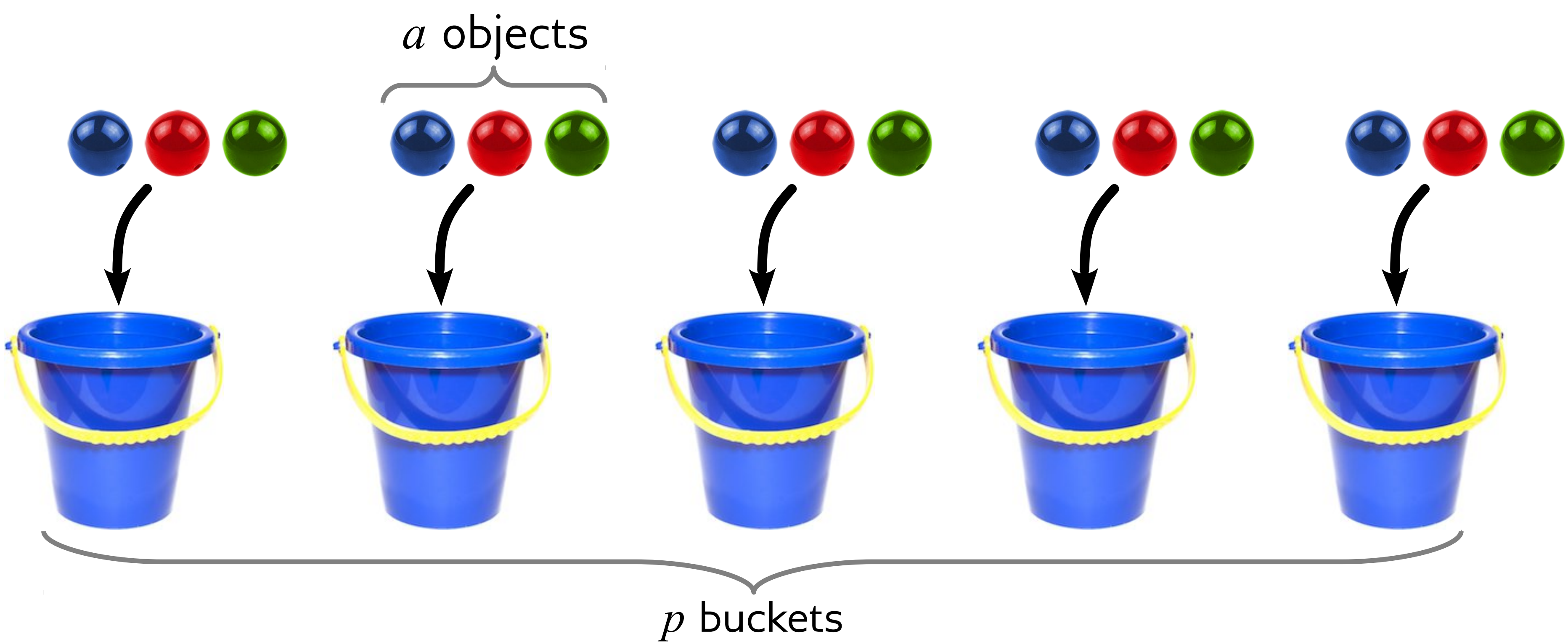
  - $123{,}456{,}789^{\,2^{57{,}885{,}161}-2} \equiv 1 \pmod{2^{57{,}885{,}161}-1}$

Two proofs

- Combinatorial
  - … counting things
- Algebraic
  - … induction
- We'll consider only non-negative a
  - … the result for non-negative a can be extended to negative integers
    (try it using what we know of congruences!)

Counting necklaces

- Due to Solomon W. Golomb, 1956
- Basic idea: a^p suggests we see how to fill p buckets, where each is filled with one of a objects



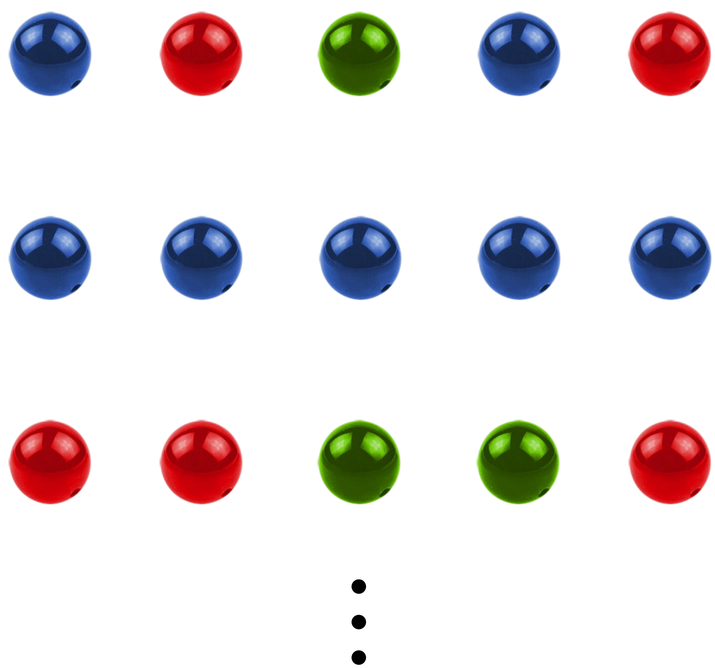
Strings of beads

- Each way of filling the buckets gives a different sequence of p objects ("beads")
  - a^p such sequences

$S_1 =$

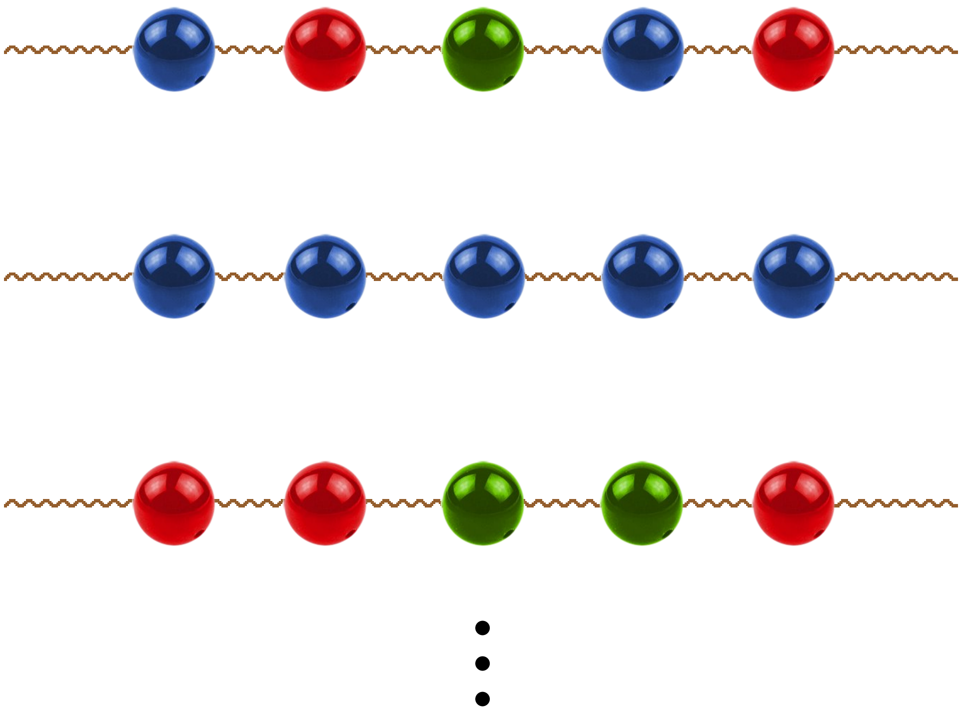
$S_2 =$

$S_3 =$



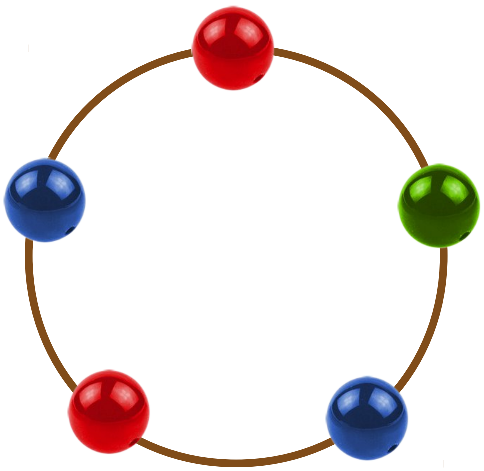
Strings of beads

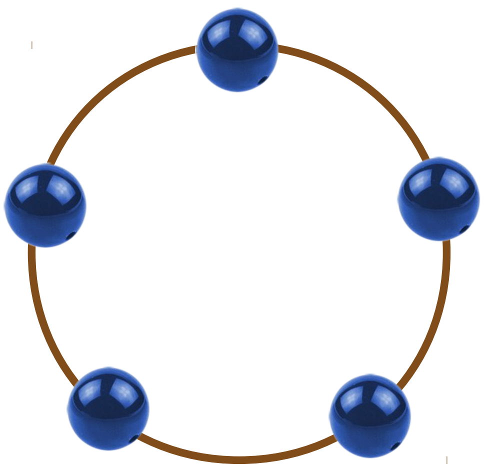
- Now string the beads together...

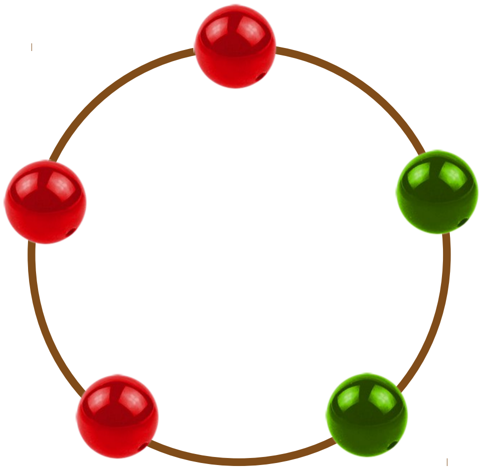


Strings of beads

- … and join the ends to form “necklaces”

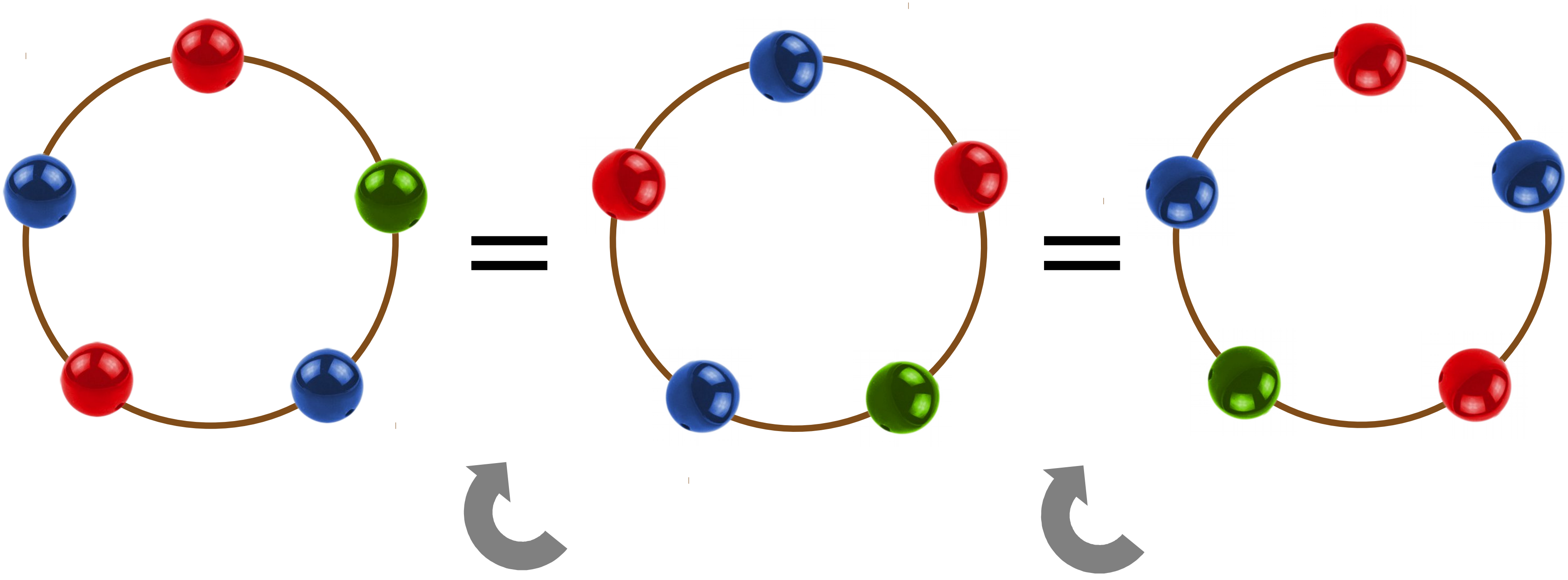






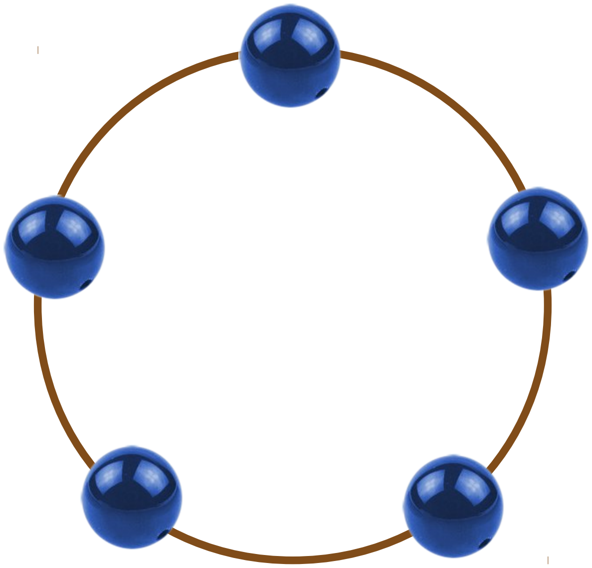
A necklace rotated…

- … is the same necklace
  - Different strings can produce the same necklace when the ends are joined



Two types of necklaces

- Containing beads of a single color



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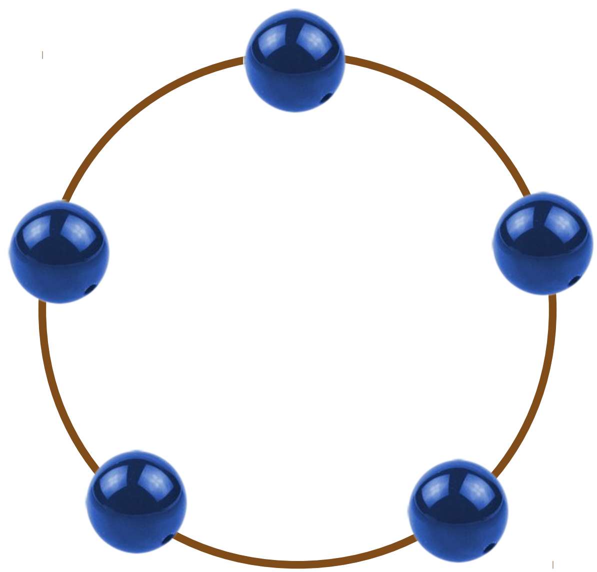
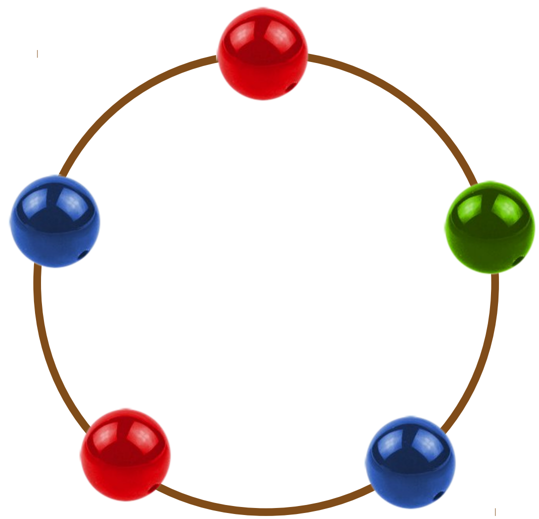


- Only one possible string

Two types of necklaces

- Containing beads of different colors



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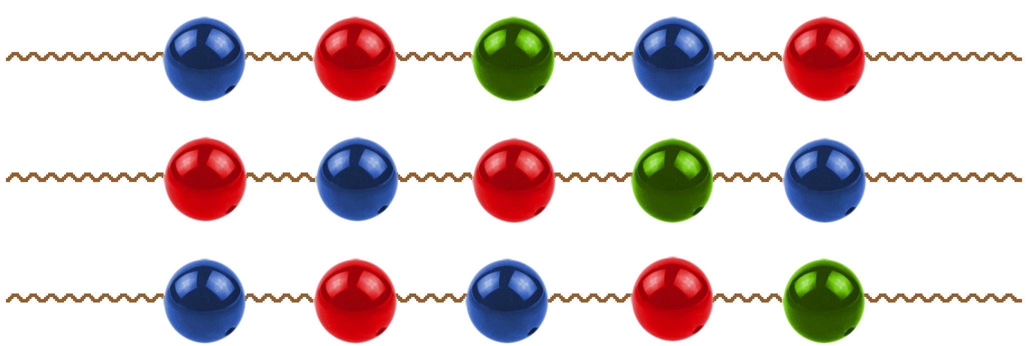
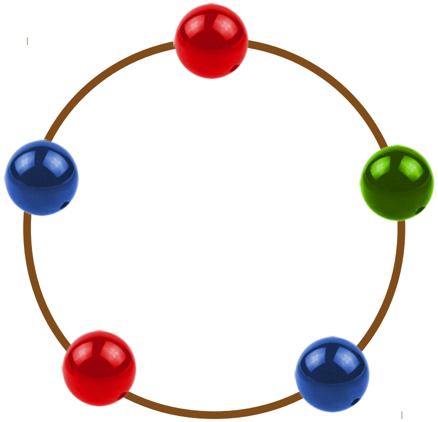
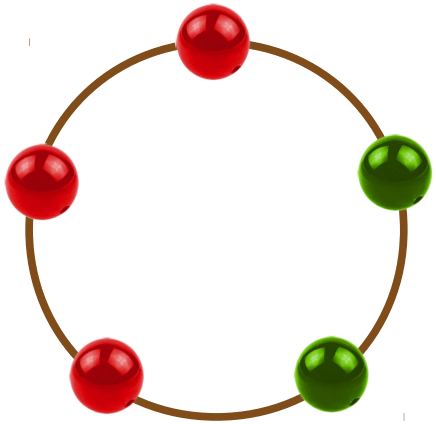


Lemma

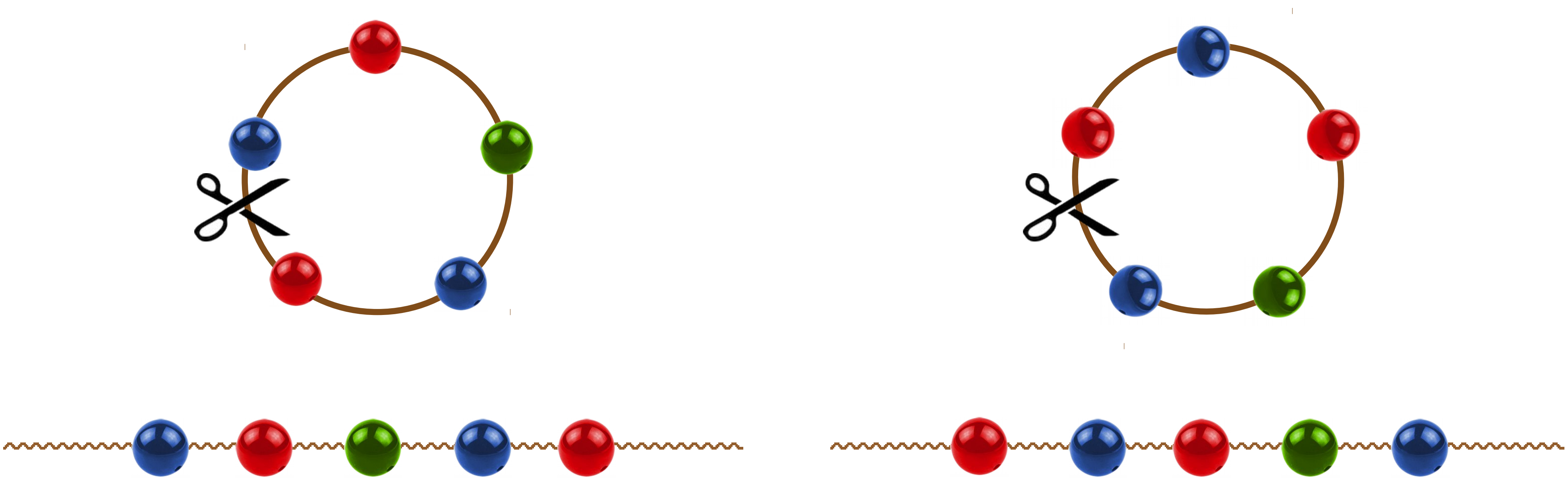
- If p is a prime number and N is a necklace with at least two colors, every rotation of N corresponds to a different string
  - … i.e. there are exactly p different strings that form the same necklace N





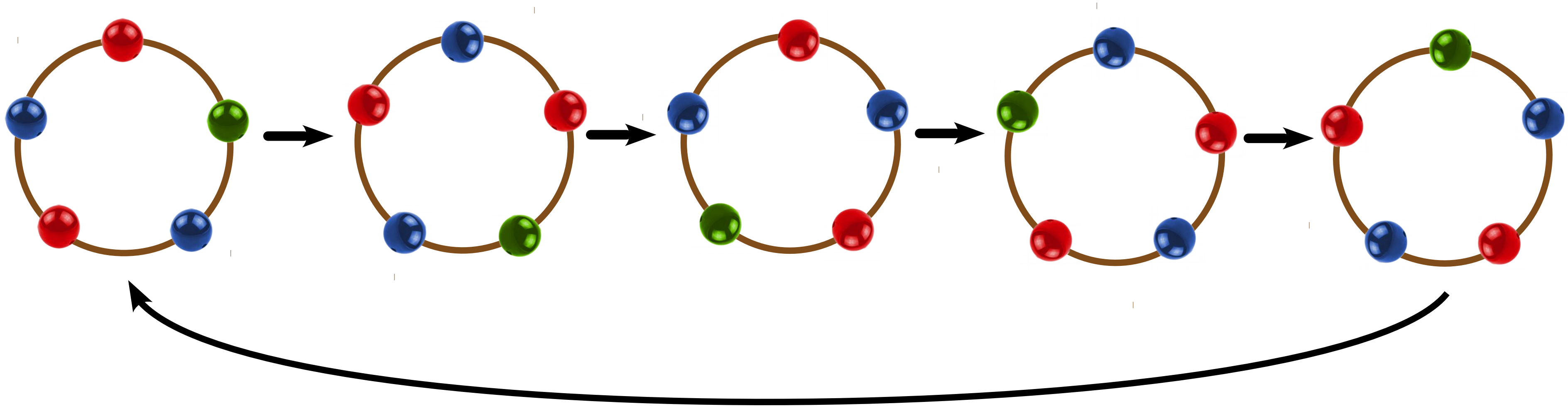
Proof of Lemma

- First, note that each string corresponds to
  - a rotation of the necklace, and then...
  - ... cutting it at a fixed point



Proof of Lemma

- No more than p strings can give the same necklace
  - There are only p (say clockwise) rotations of the necklace (that align the beads) before we loop back to the original orientation



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  - ... which is a contradiction, unless $r = 0$

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- This proves the lemma

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$\Rightarrow a^p \equiv a \pmod{p}$ QED!

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- By the Binomial Theorem,

$$(a+1)^p = a^p + \binom{p}{1} a^{p-1} + \binom{p}{2} a^{p-2} + \binom{p}{3} a^{p-3} + \ldots + \binom{p}{p-1} a + 1$$

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Binomial coefficient $\binom{p}{k}$ is $p!/k!(p-k)!$, which is always an integer. p is prime, so it isn't canceled out by terms in the denominator

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- Hence proved by induction