Kleene's Theorem

CS 2800: Discrete Structures, Fall 2014

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  - Regex has FA
    - Relatively simple construction
  - FA has regex
    - Tricky to prove

Regex → FA

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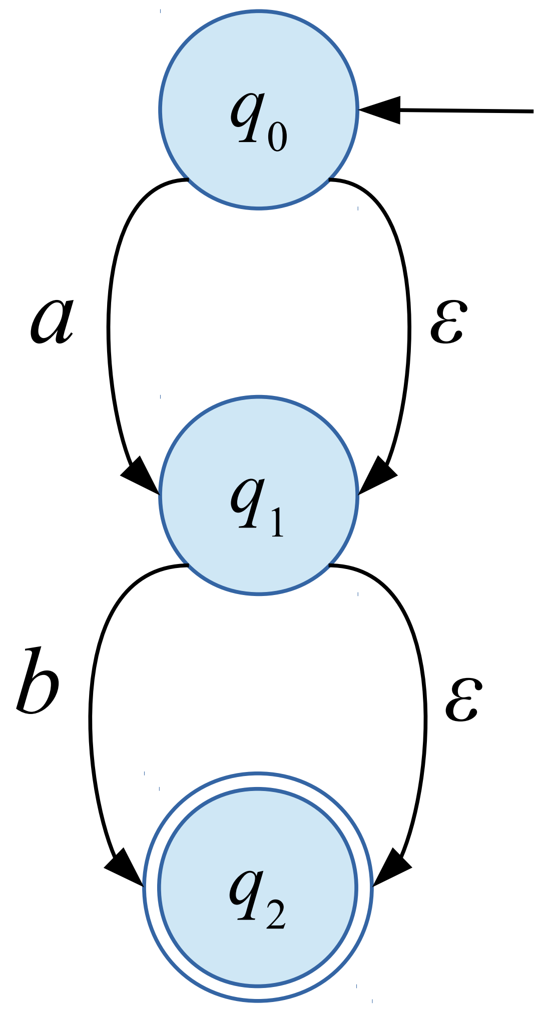
  - … which can be converted to an NFA
  - … which can be converted to a DFA

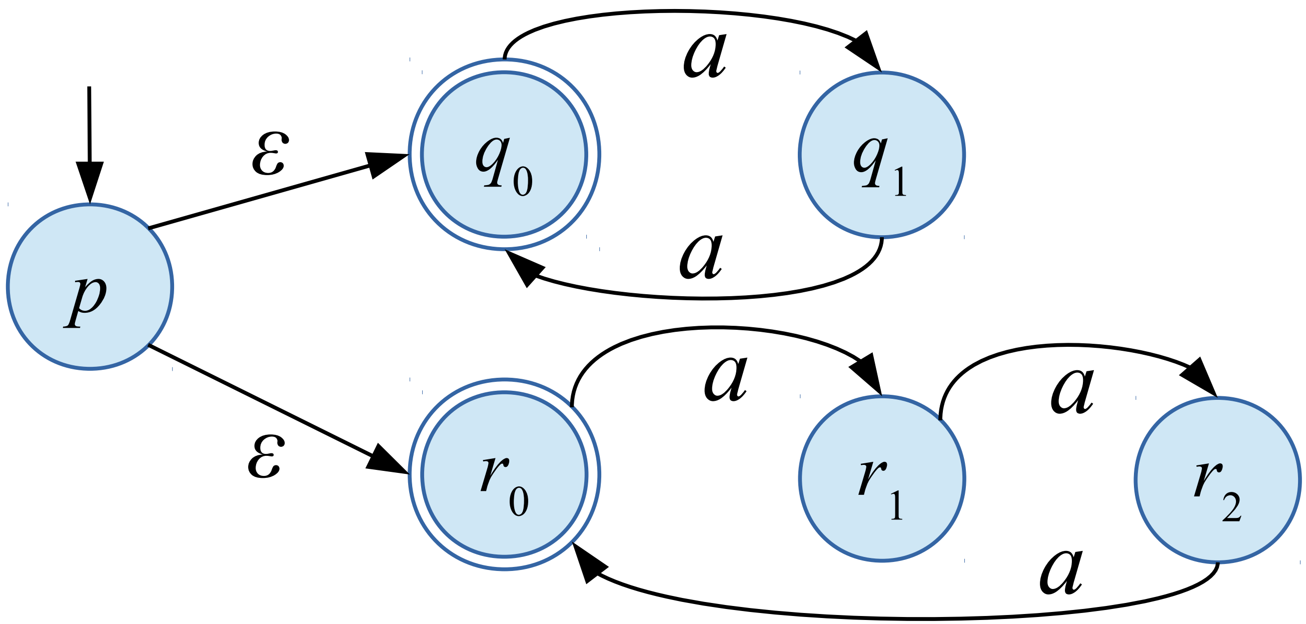
Recap: NFAs with epsilon transitions

- Just like ordinary NFAs, but...
  - Can "instantaneously" change state *without* reading an input symbol
  - Valid transitions of this type are shown by arcs labeled 'ε'
  - Note that ε does not suddenly become a member of the alphabet. Instead, we assume ε does not belong to *any* alphabet – it's a special symbol.



Why ε-NFAs?

- Suitable for representing "or" relations
- E.g. $L = \{\ a^n \mid n \in \mathbf{N} \text{ is divisible by 2 or 3 }\}$



- … but they're equivalent to NFAs and DFAs

ε-NFA to ordinary NFA

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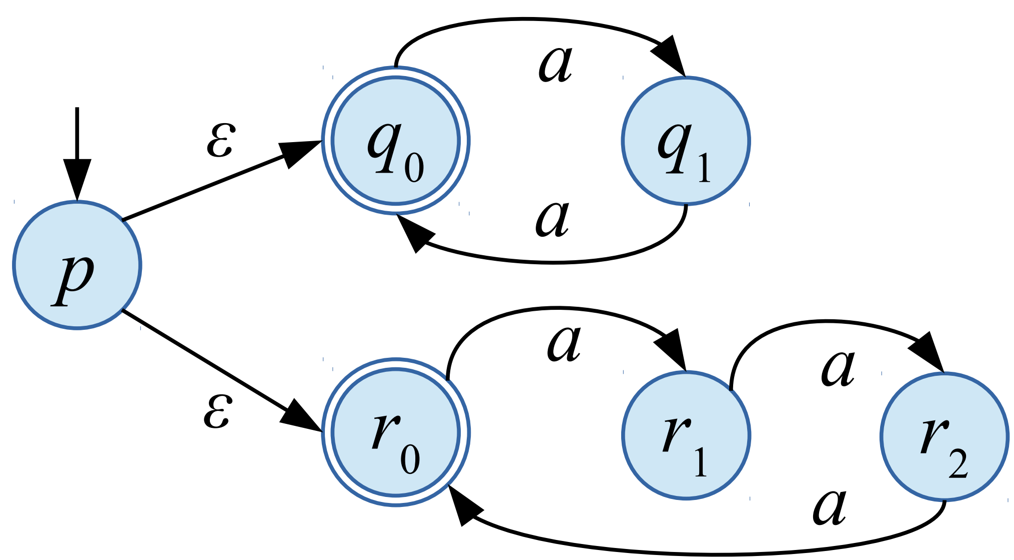
$$ECLOSE(p) = ?$$



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  - The set includes q itself
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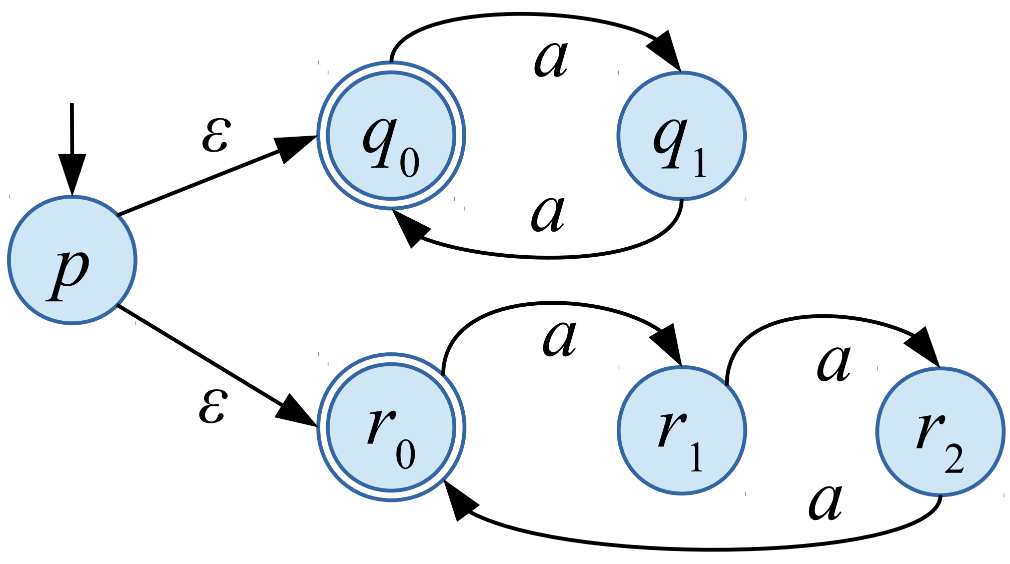
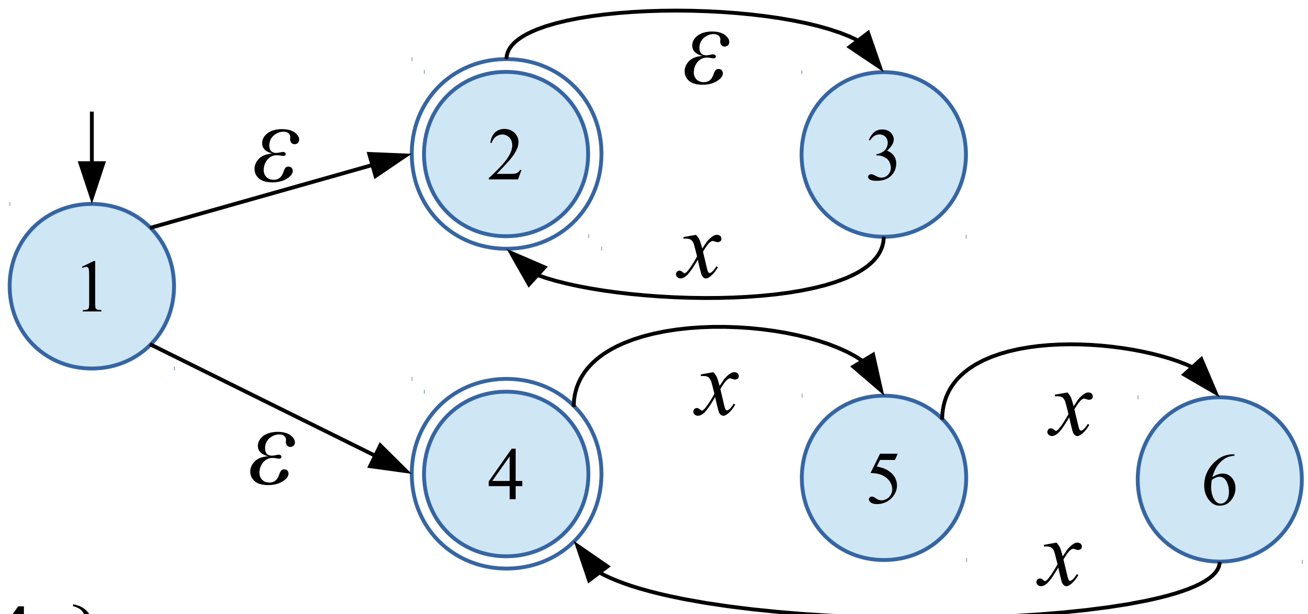
$$ECLOSE(p) = \{\, p, q_0, r_0 \,\}$$



$ECLOSE(1)$?



A) { 2, 4 }
B) { 1, 2, 4 }
C) { 1, 2, 3, 4 }
D) { 2, 3, 4 }

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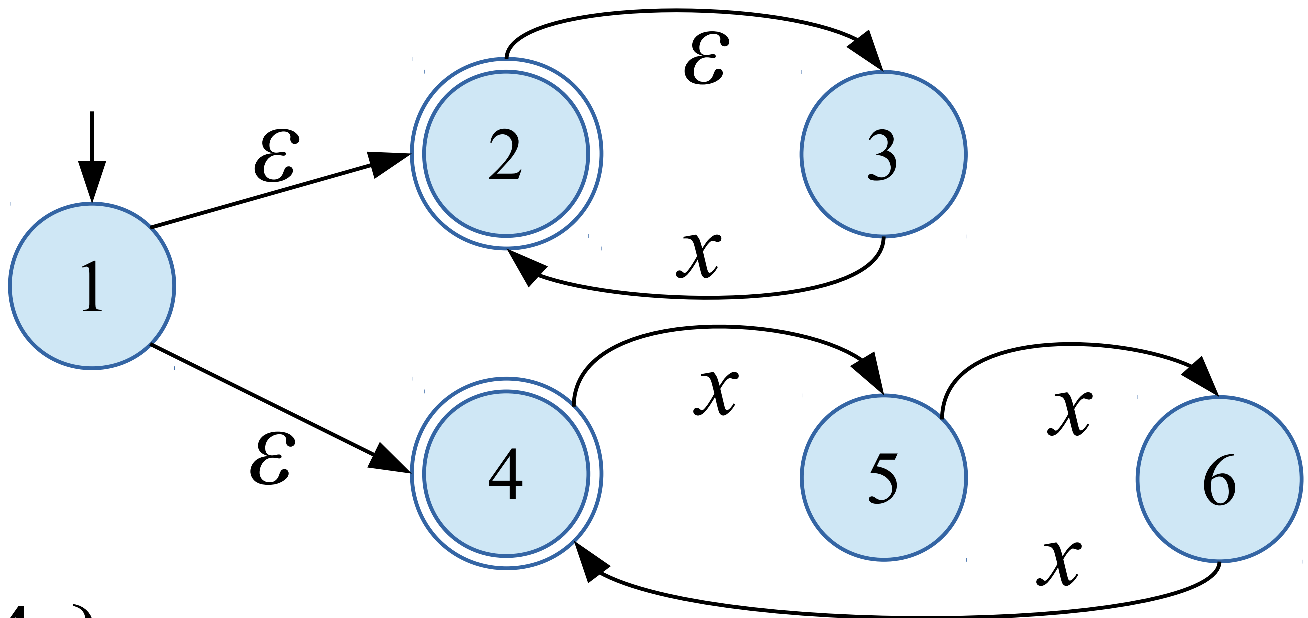

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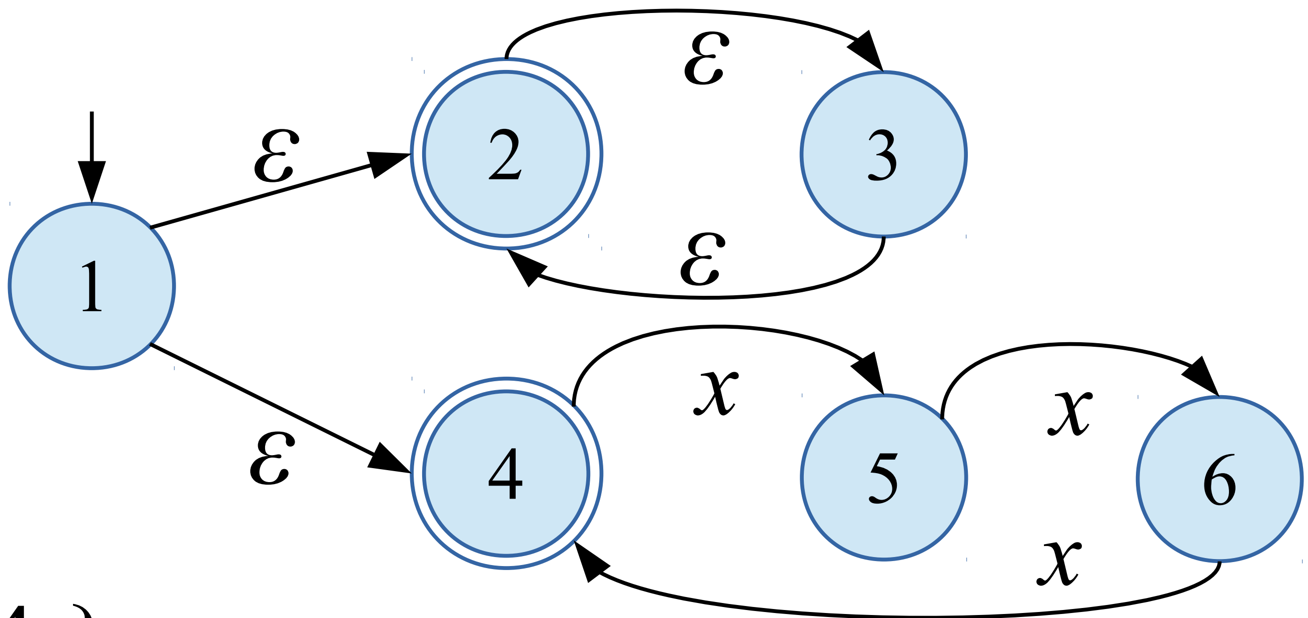

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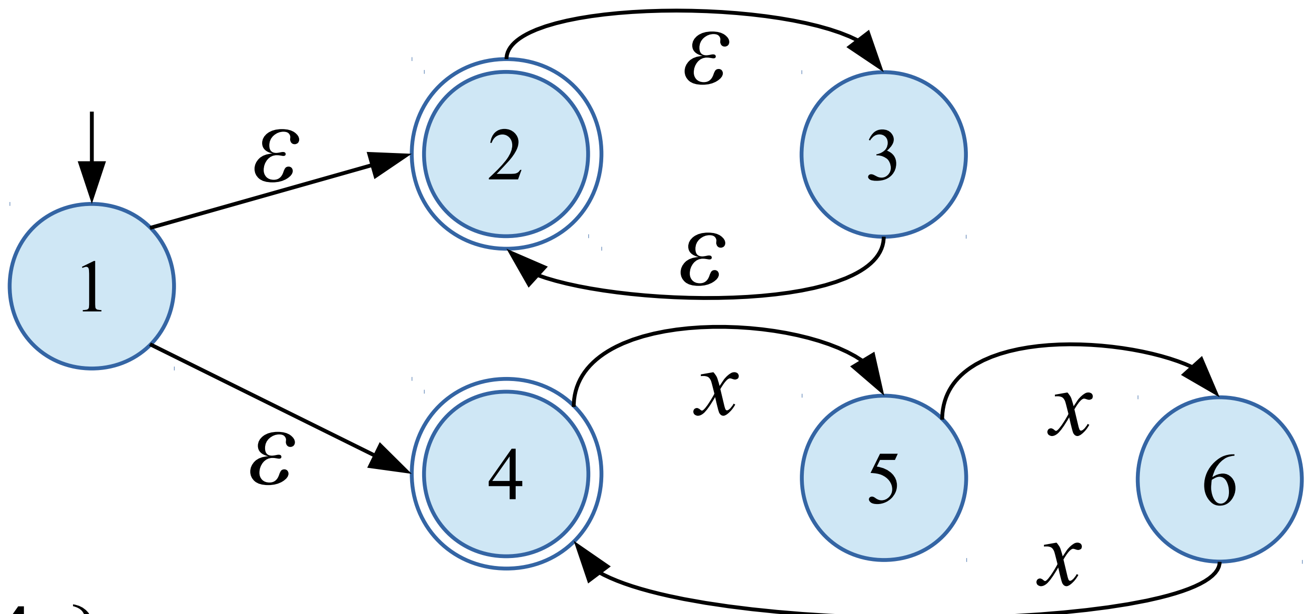

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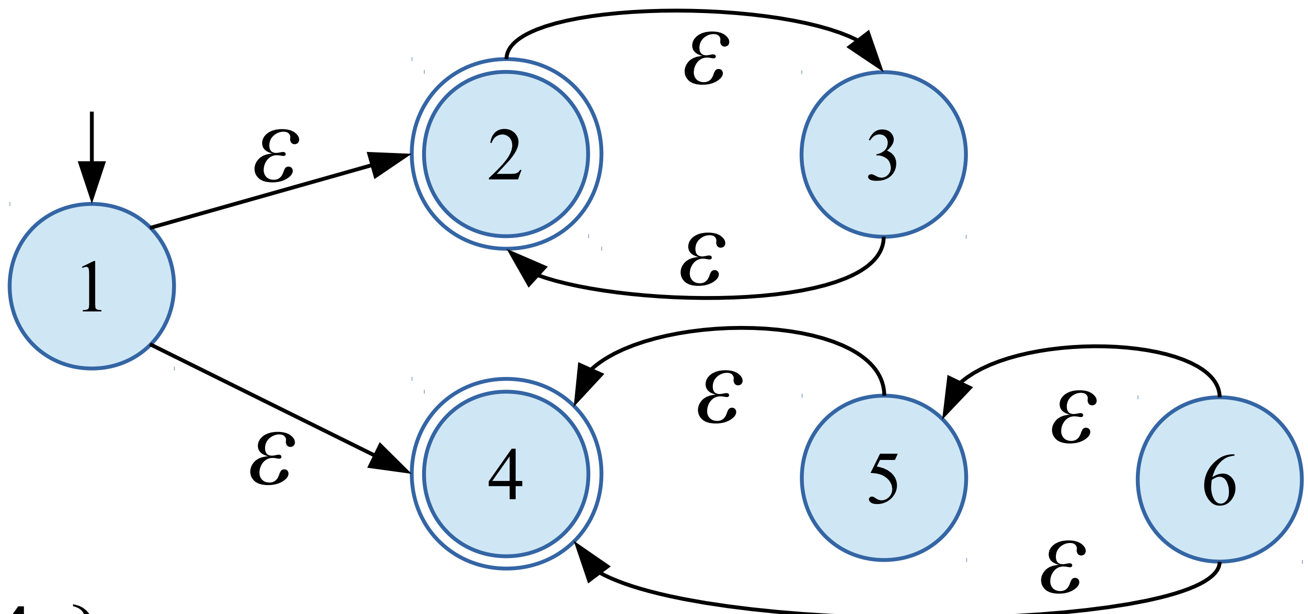

A) $\{ 2, 4 \}$
B) $\{ 1, 2, 4 \}$
C) $\{ 1, 2, 3, 4 \}$
D) $\{ 1, 2, 3, 4, 5, 6 \}$

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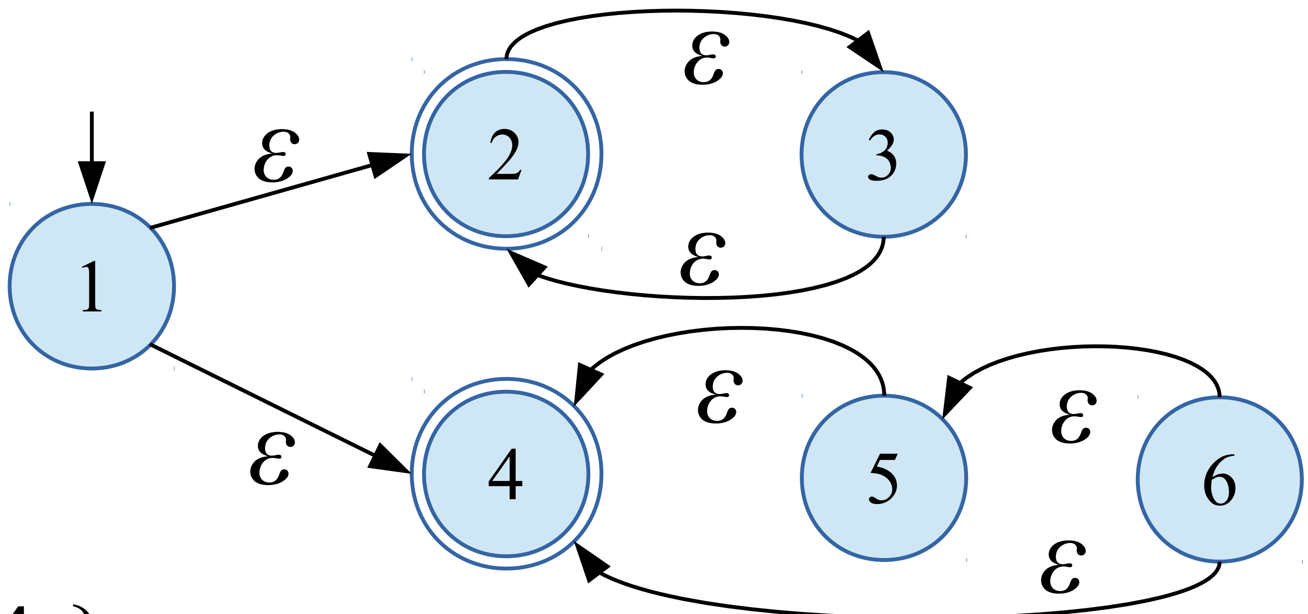

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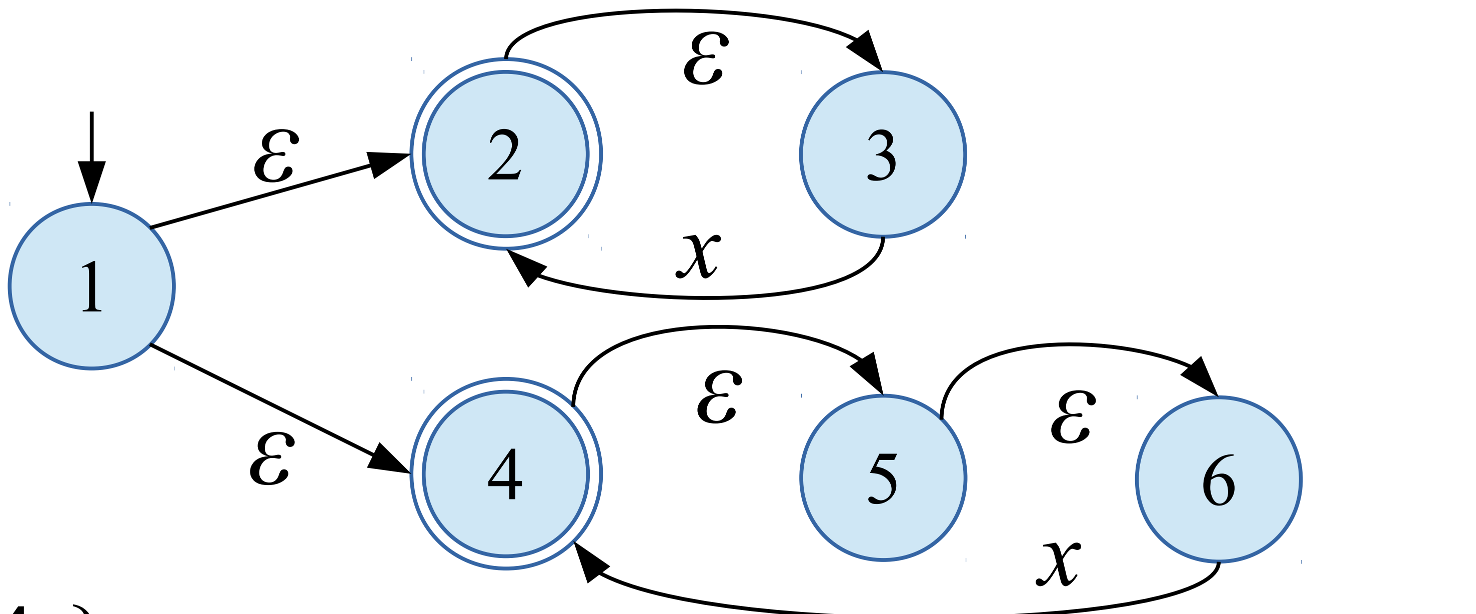

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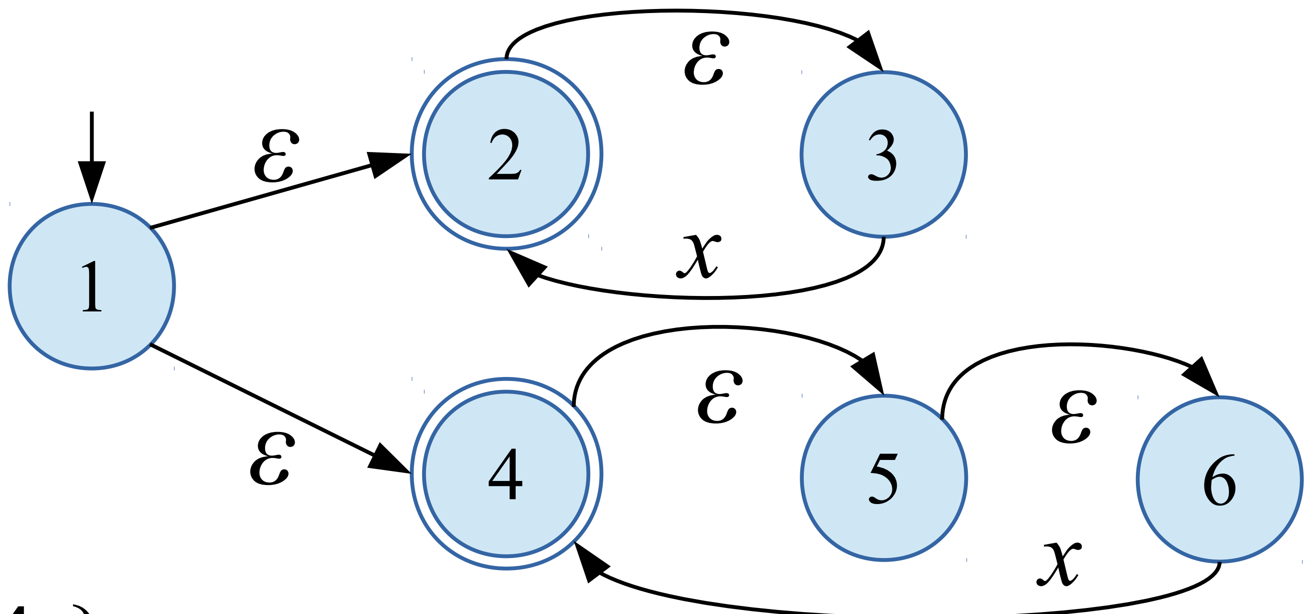

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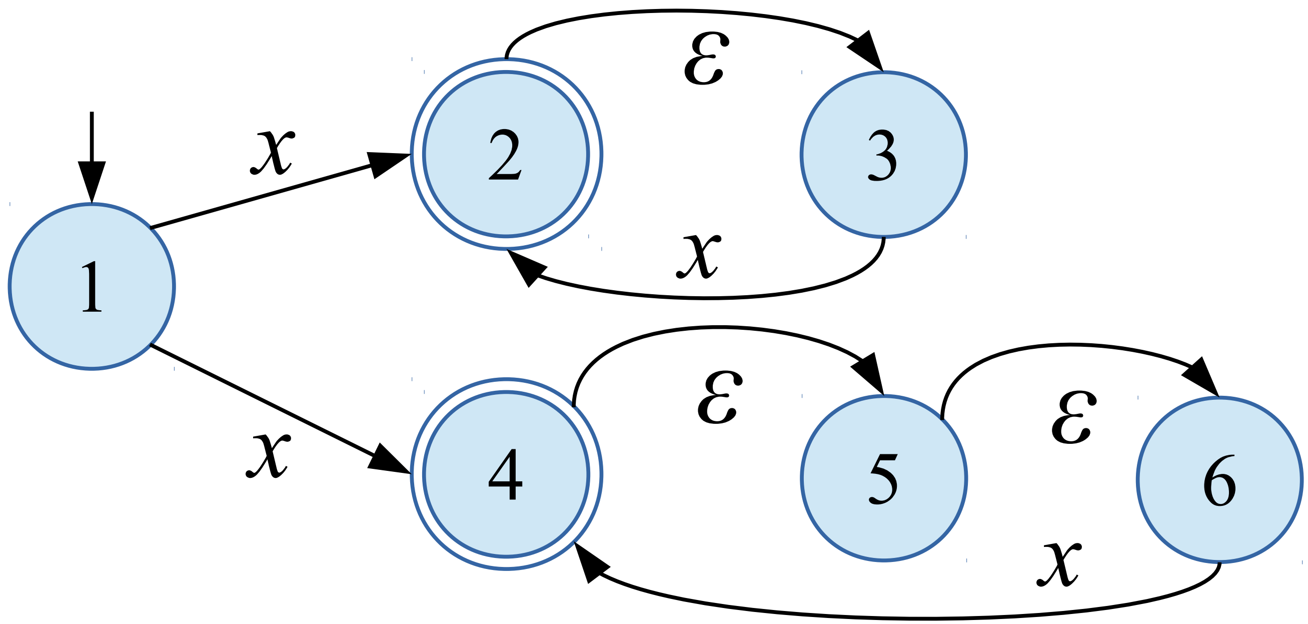

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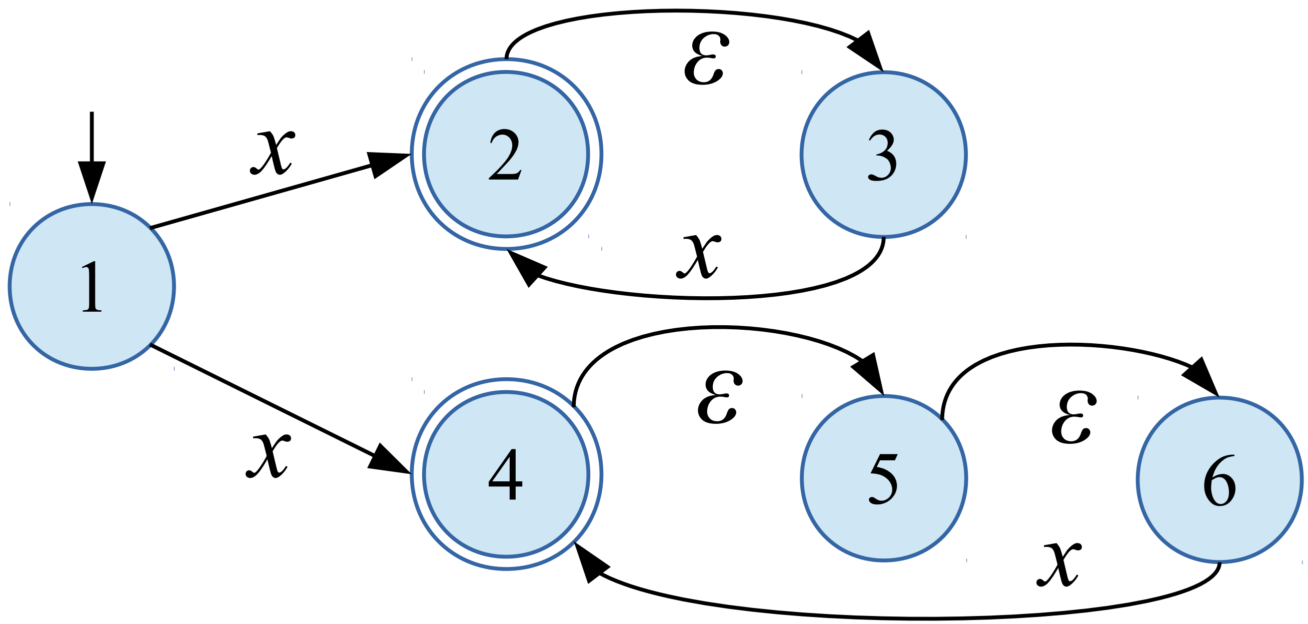

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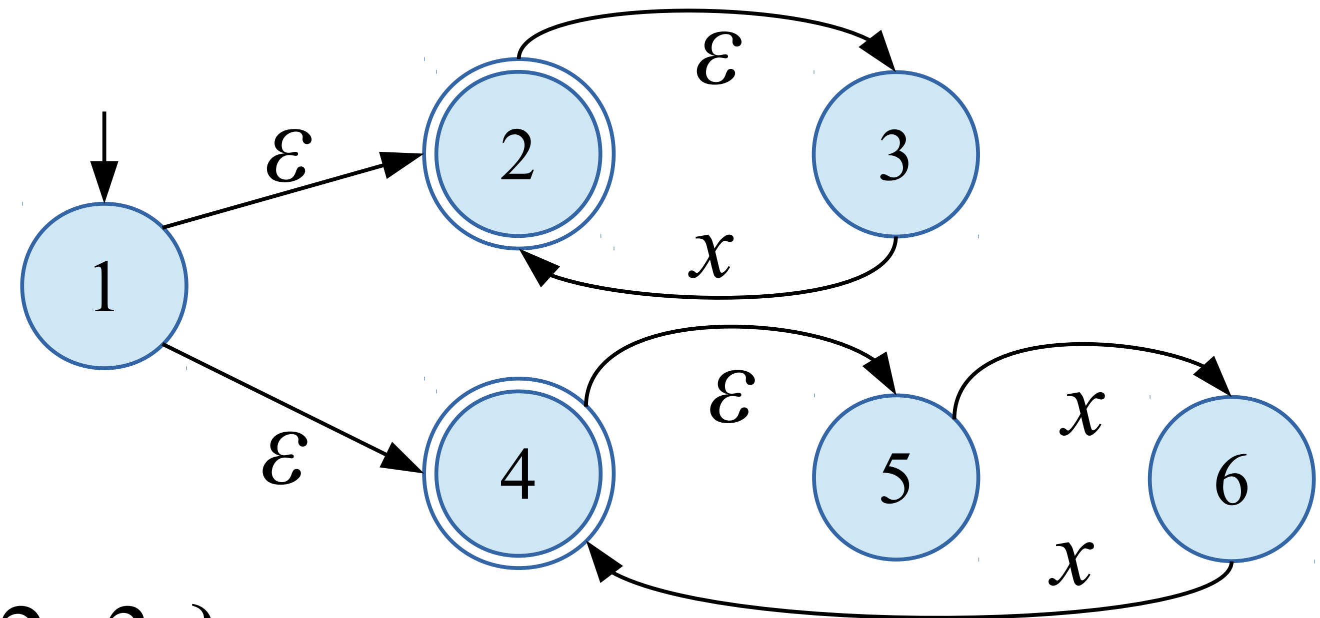

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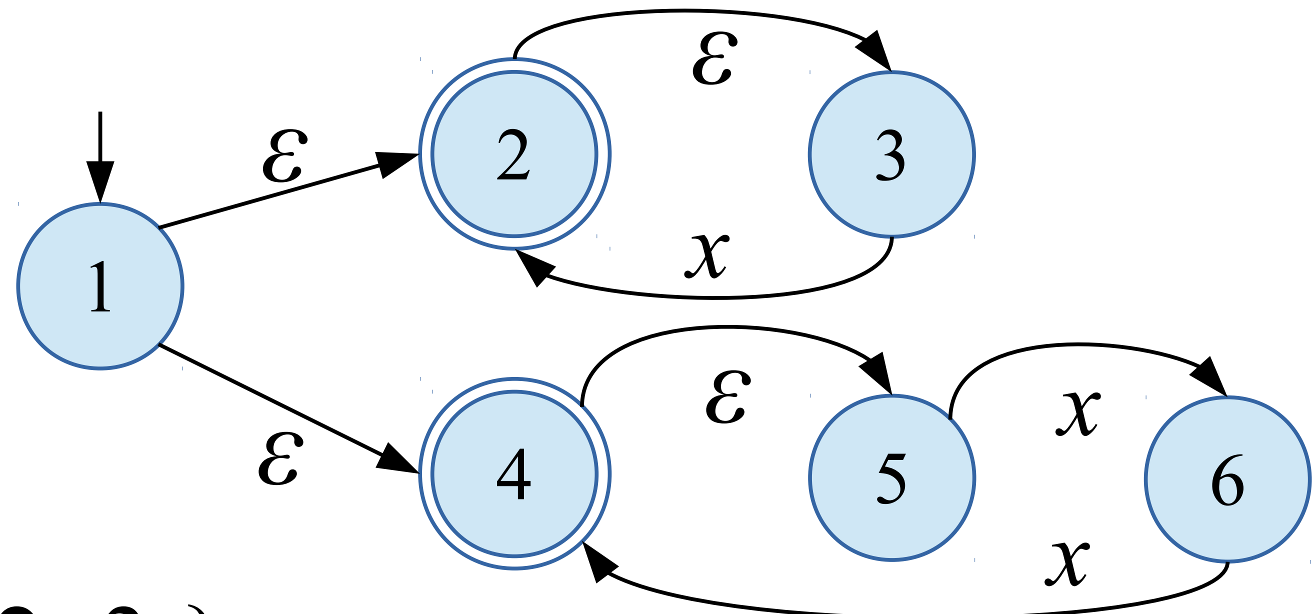

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E) **{ 1, 2, 3, 4, 5 }**

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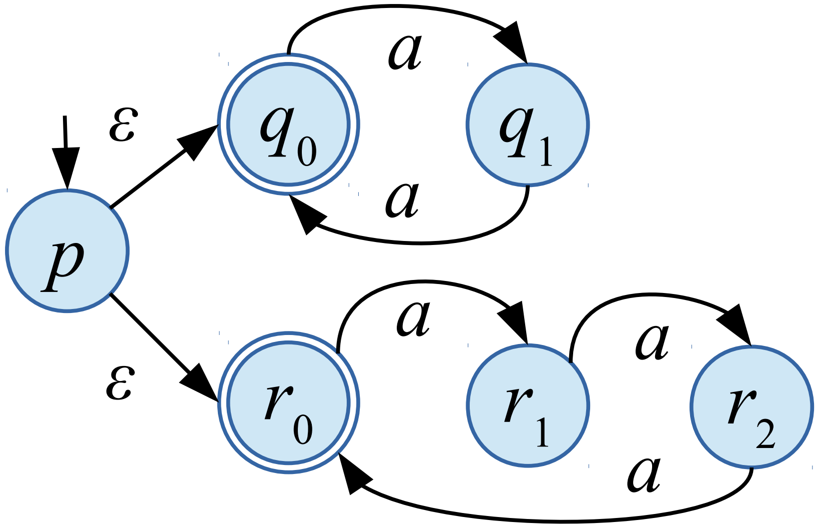
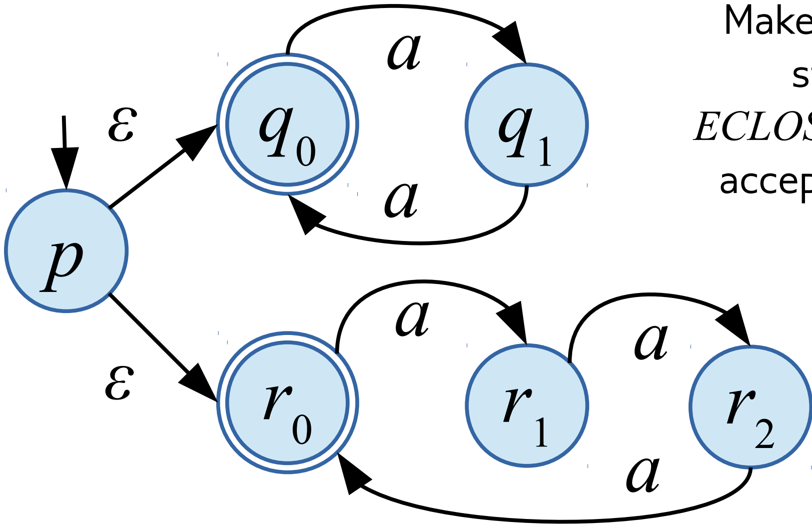

ε
ε
p
q0
q1
r0
r1
r2
a
a
a
a
a



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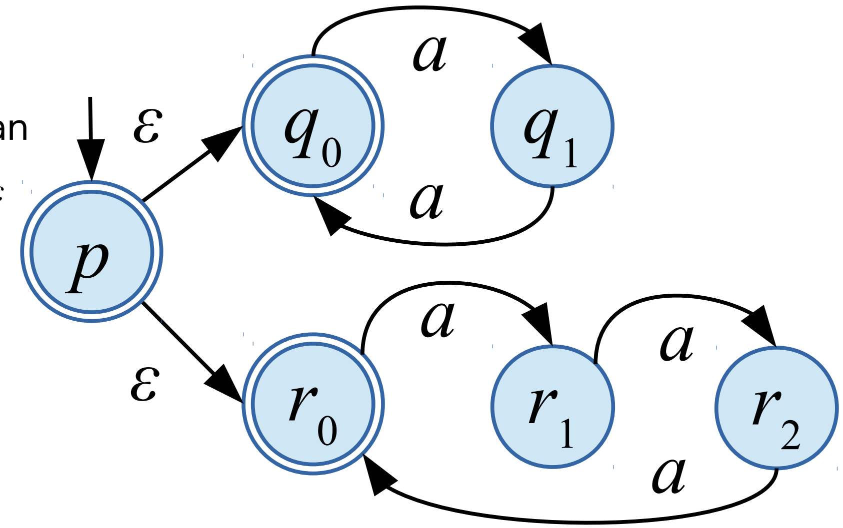

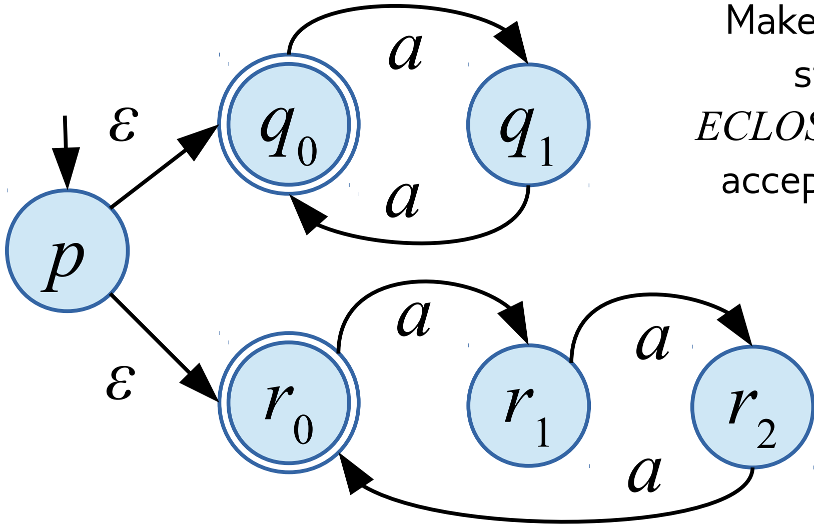

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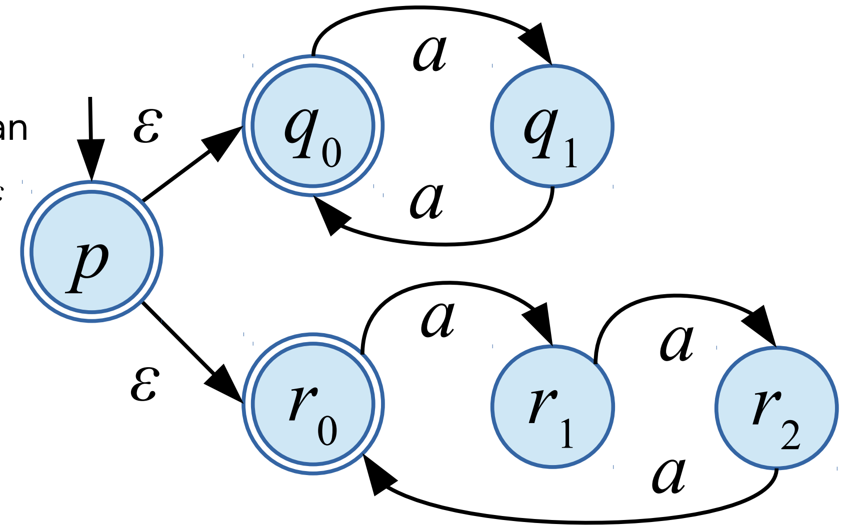

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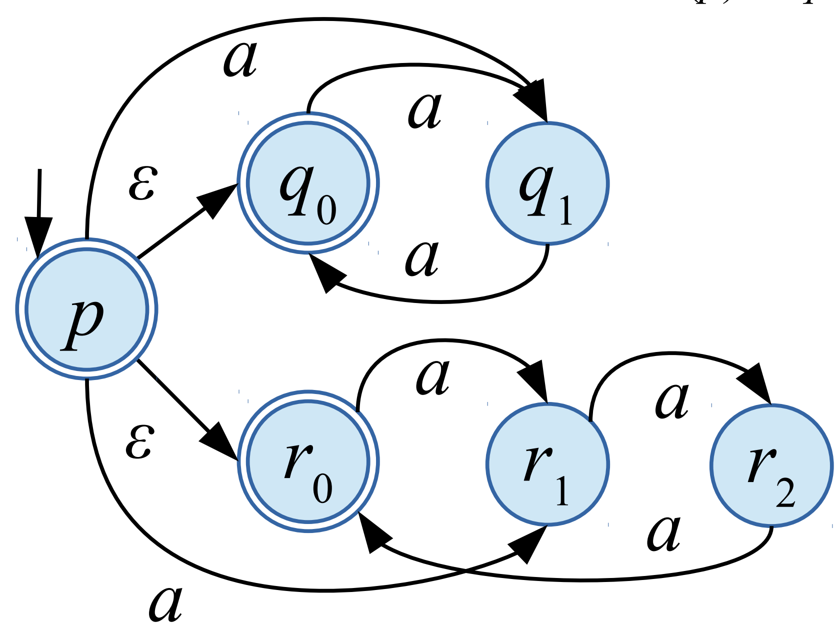

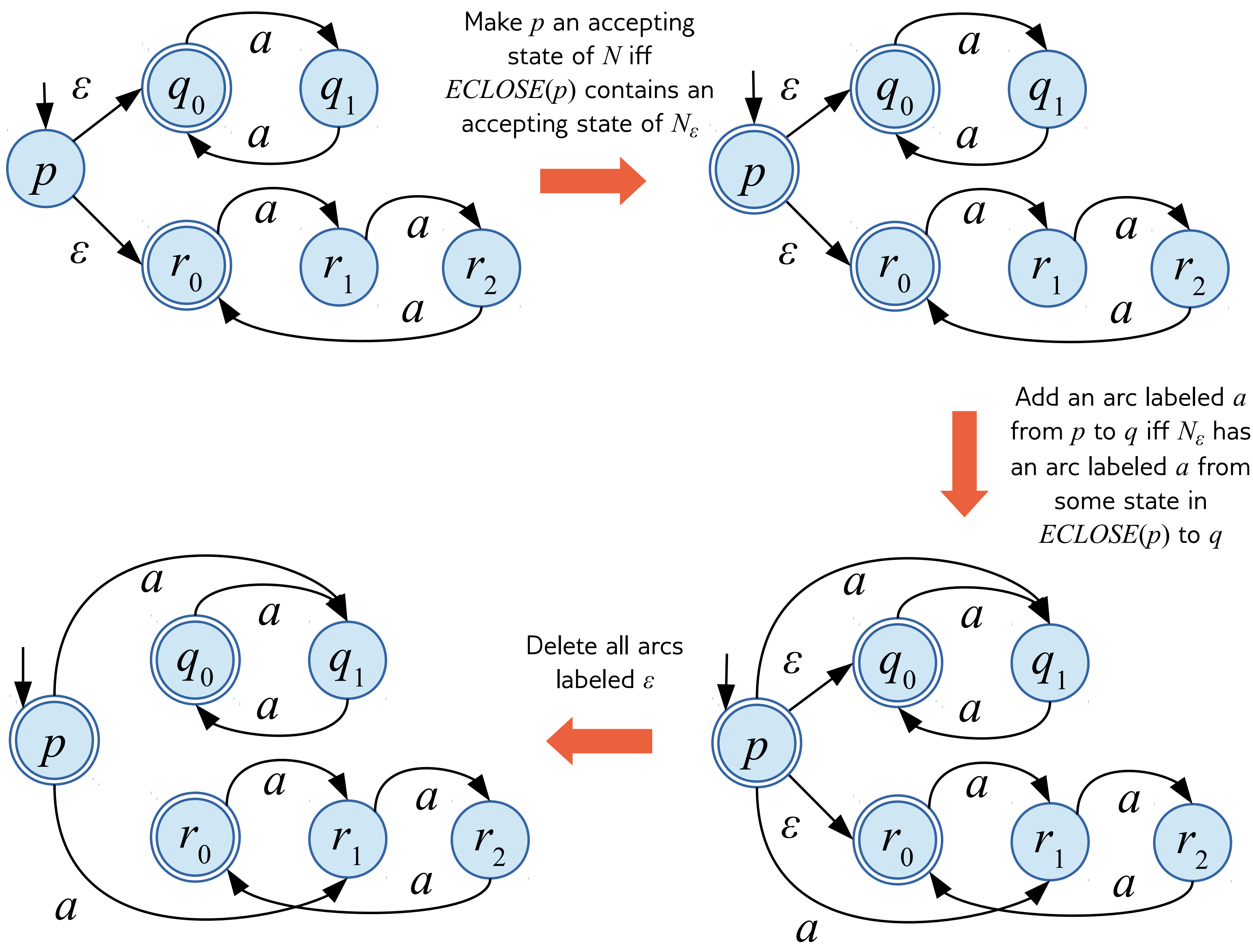

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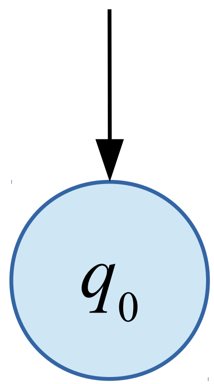
Regular expression to ε-NFA

- Structural induction on regex
  - Construct simple automata for base cases
  - For every higher-order construction, construct equivalent ε-NFA from smaller ε-NFAs

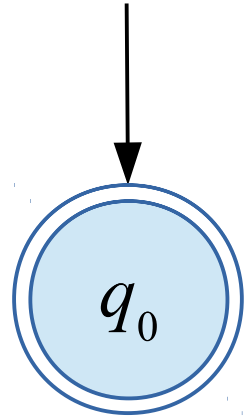
Empty set

Regex: $\varnothing$



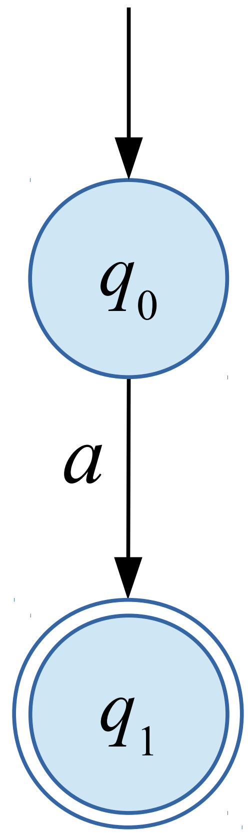
Empty string

Regex: $\boldsymbol{\varepsilon}$



Literal character

Regex: **a**



Concatenation

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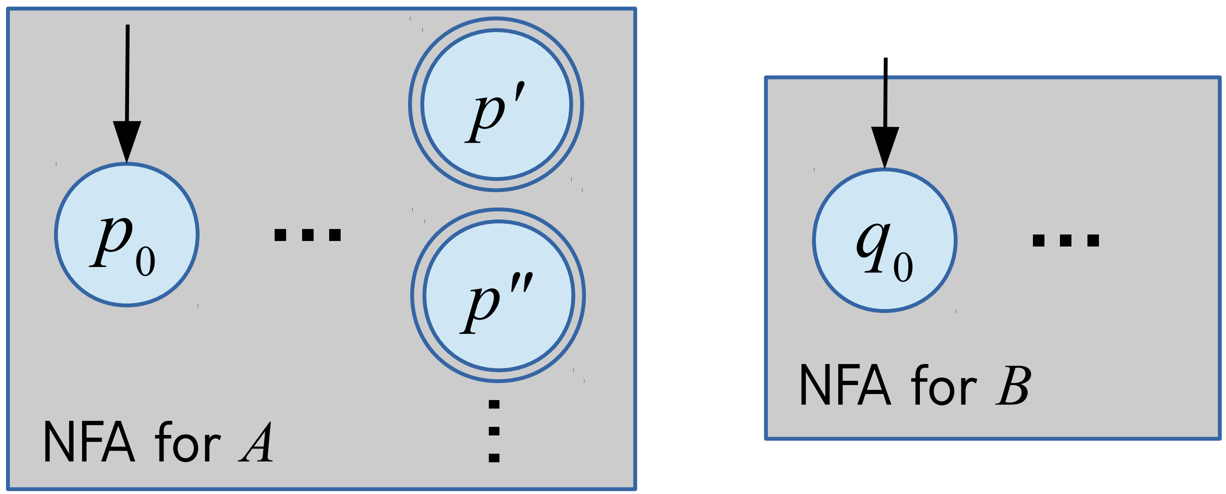

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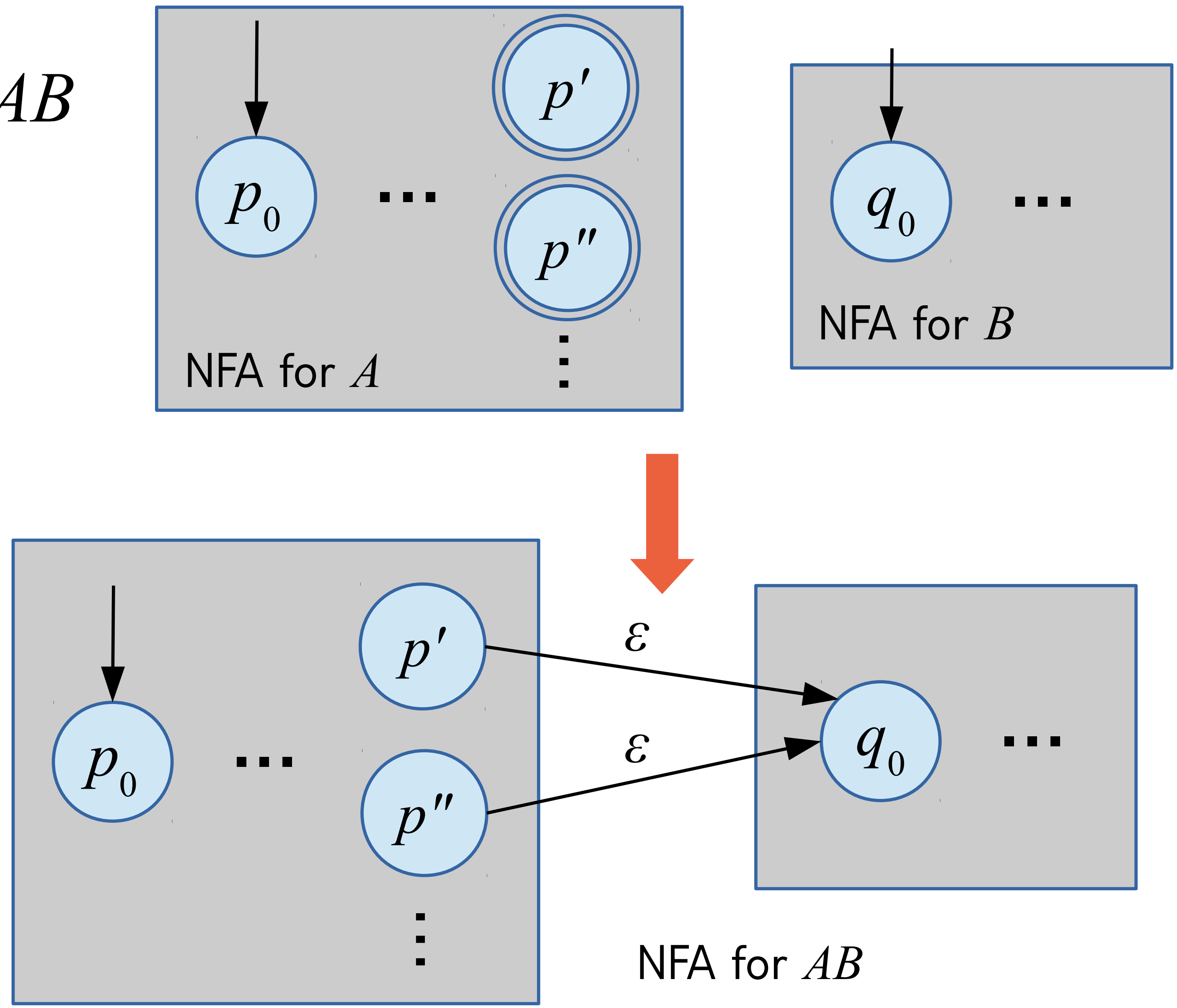

Alternation

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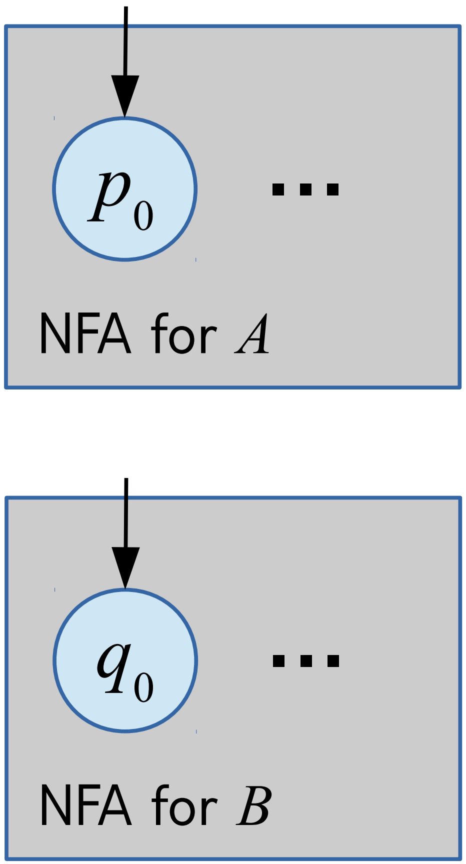

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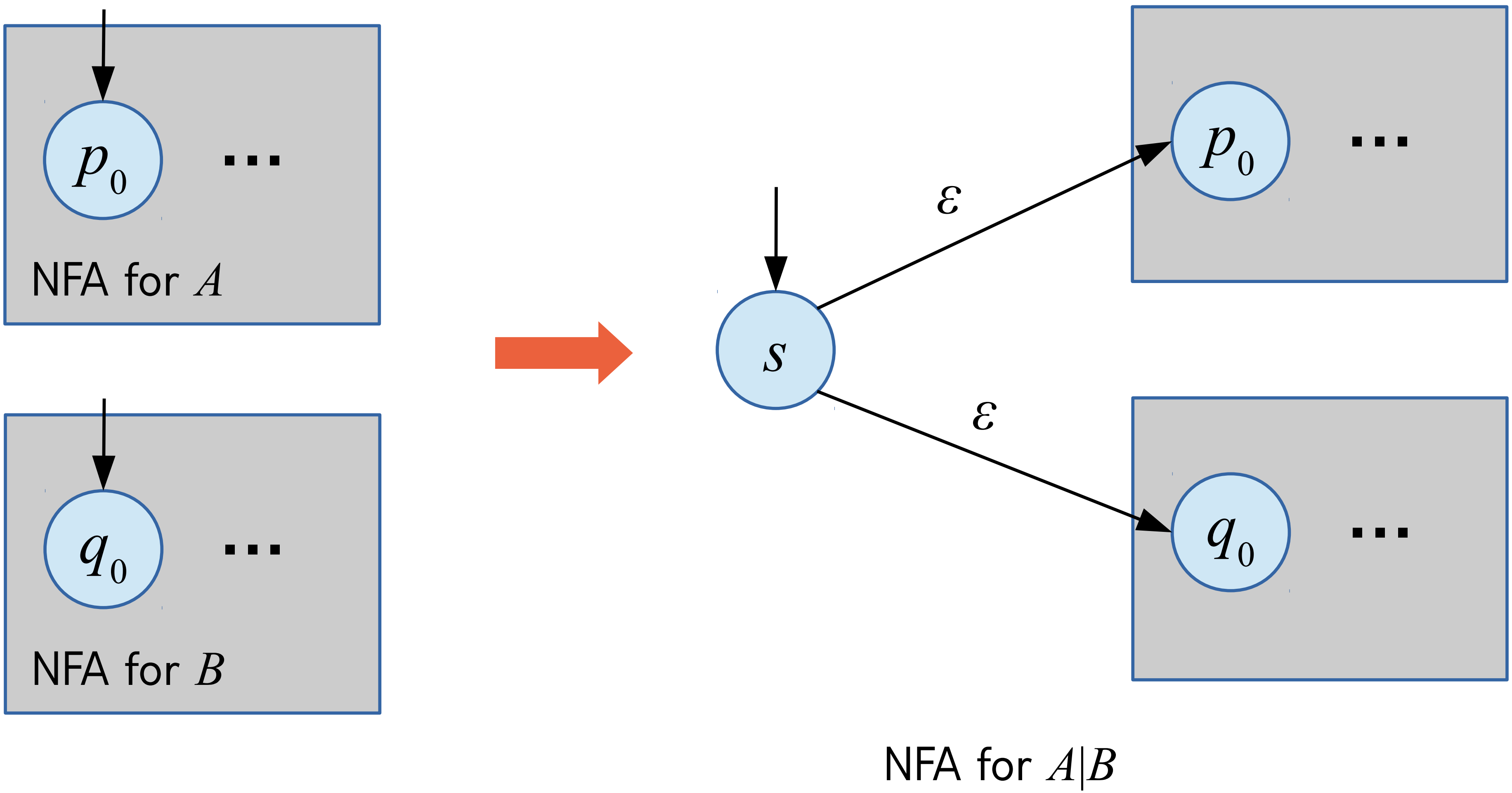

Kleene star

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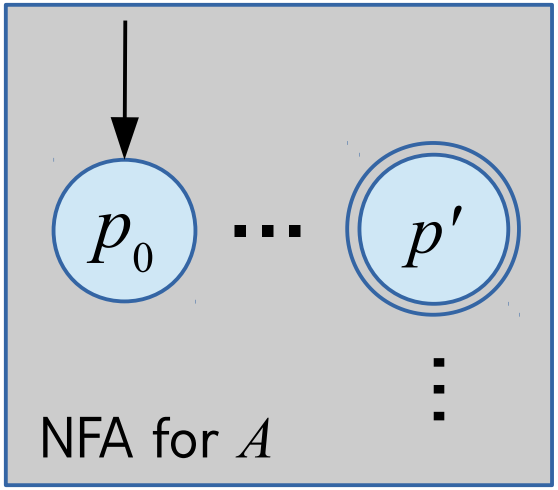

Kleene star

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