Regular Expressions

CS 2800: Discrete Structures, Fall 2014

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Regular expressions ("regex"-es) are defined *inductively*

(start with simple base expressions, construct more complicated ones recursively)

Empty set

$$L(\emptyset) = \emptyset$$

Empty string

$$L(\boldsymbol{\varepsilon}) = \{\varepsilon\}$$

Literal character

$$L(\mathbf{x}) = \{x\}$$

e.g. $L(\mathbf{1}) = \{1\}, L(\mathbf{2}) = \{2\}, L(\mathbf{a}) = \{a\}$

Concatenation

$$L(AB) = \{ab \mid a \in L(A), \mathrm{b} \in L(B)\}$$

e.g. $L(\mathbf{12}) = \{12\}, L(\mathbf{aabb}) = \{aabb\}$

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Kleene star

$$L(A^*) =$$
$$\{\varepsilon\} \cup \{\, x_1 x_2 \ldots x_n \mid n \in \mathbf{N},\, x_i \in L(A) \,\}$$

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Playing with regexes

- http://regex101.com/
- http://rubular.com/
- http://www.google.com/search?q=online+regex+tester

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    - Relatively simple construction
  - FA has regex
    - Tricky to prove

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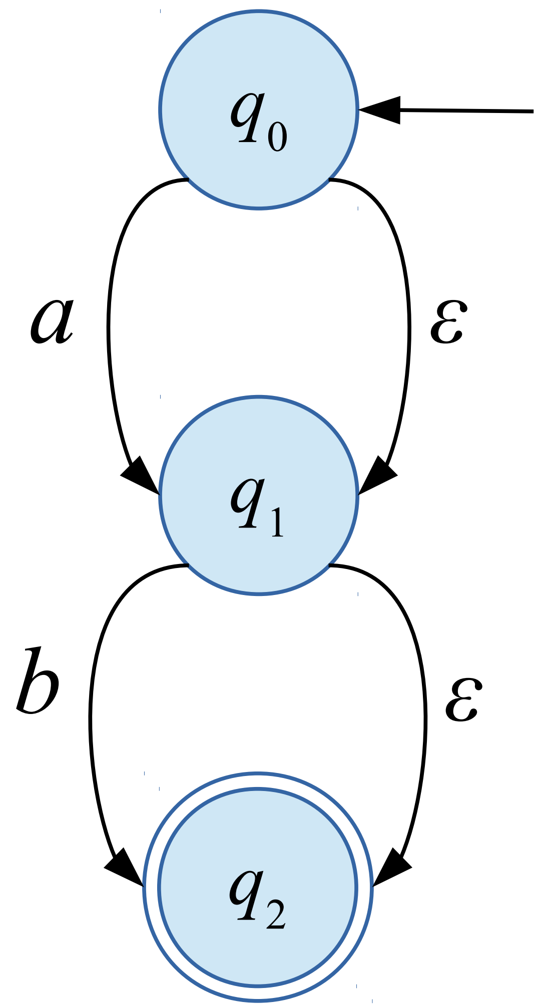
  - … which can be converted to an NFA
  - … which can be converted to a DFA

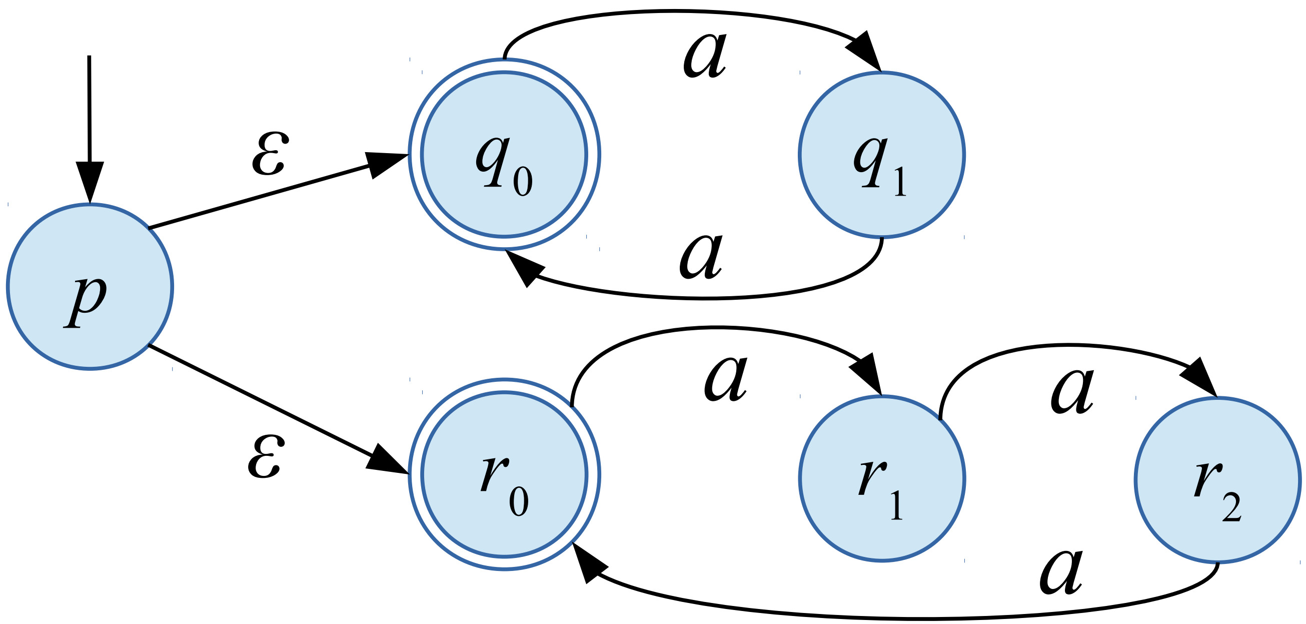
Recap: NFAs with epsilon transitions

- Just like ordinary NFAs, but…
  - Can "instantaneously" change state *without* reading an input symbol
  - Valid transitions of this type are shown by arcs labeled 'ε'
  - Note that ε does not suddenly become a member of the alphabet. Instead, we assume ε does not belong to *any* alphabet – it's a special symbol.



Why ε-NFAs?

- Suitable for representing "or" relations
- E.g. $L = \{ \, a^n \mid n \in \mathbf{N} \text{ is divisible by 2 or 3} \, \}$



- ... but they're equivalent to NFAs and DFAs