Finite Automata (contd)

CS 2800: Discrete Structures, Fall 2014

Sid Chaudhuri

Recap: Deterministic Finite Automaton

- A DFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$
  - Q is a finite set of states
  - Σ is a finite input alphabet (e.g. $\{0, 1\}$)
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  - $q_0 \in Q$ is the start/initial state
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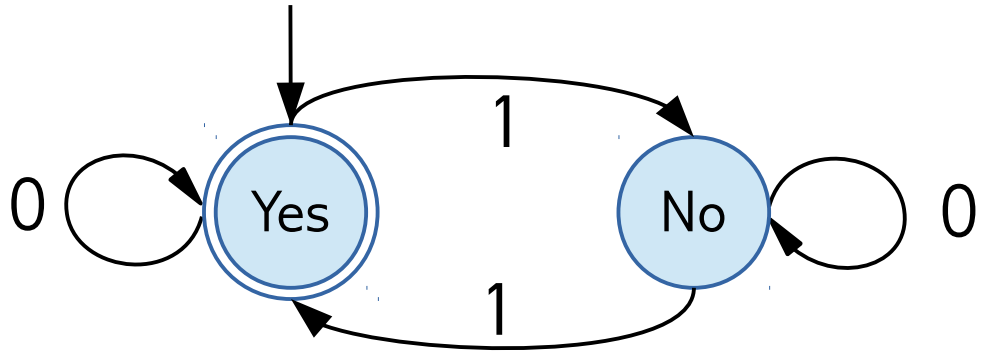
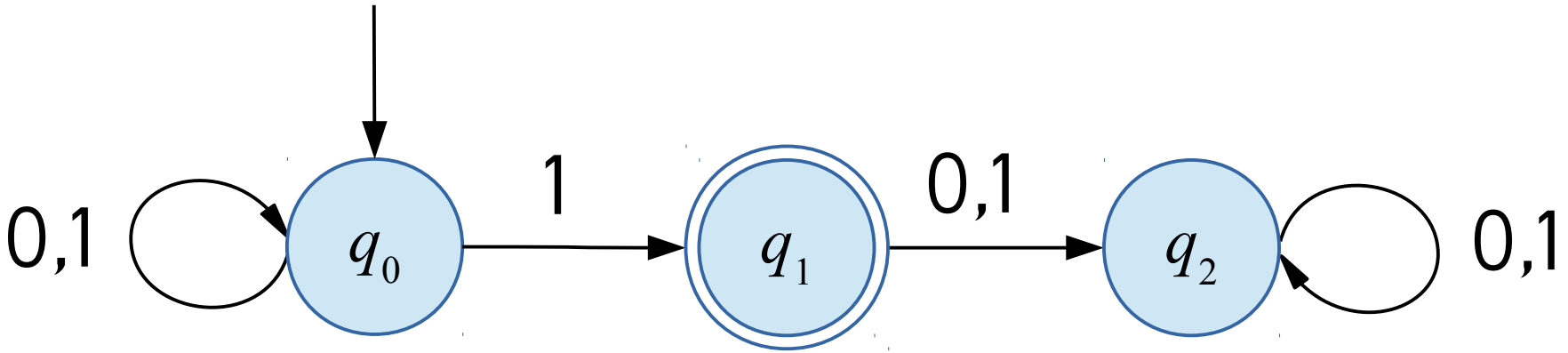


A ***non***-deterministic finite automaton



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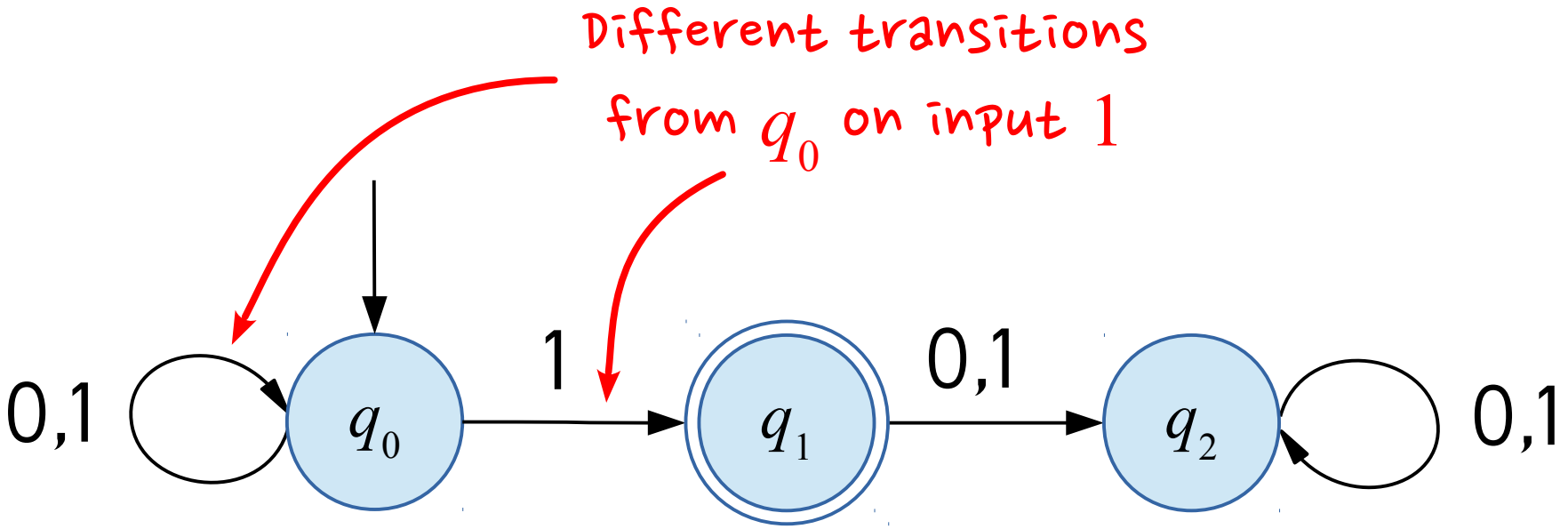

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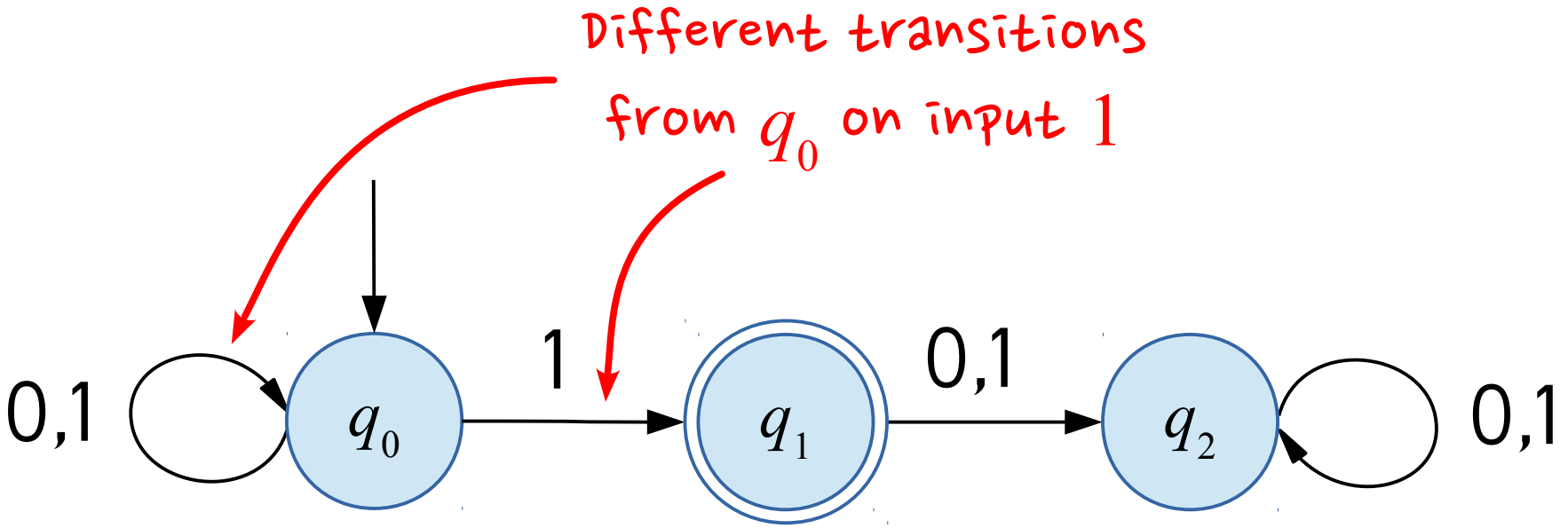

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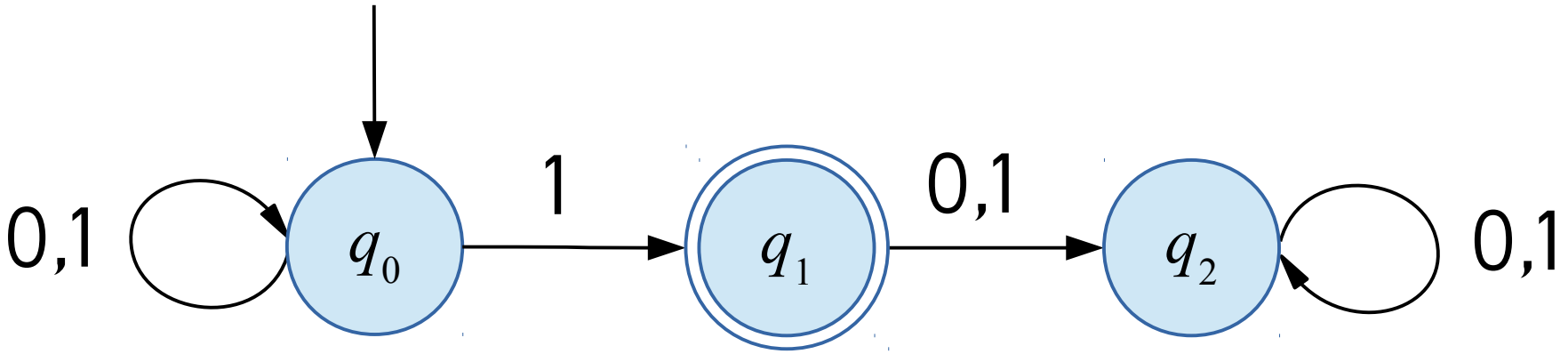

What language does this automaton accept?

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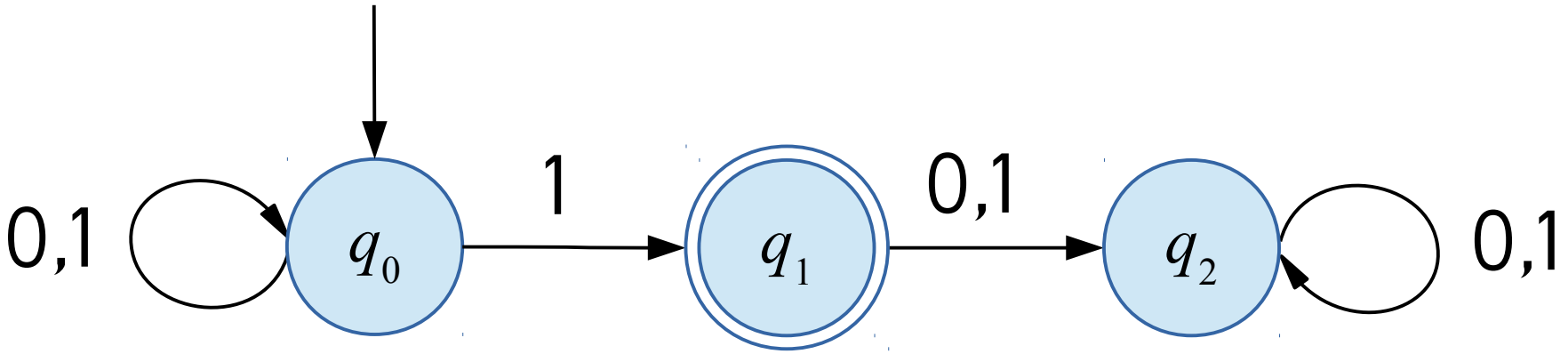
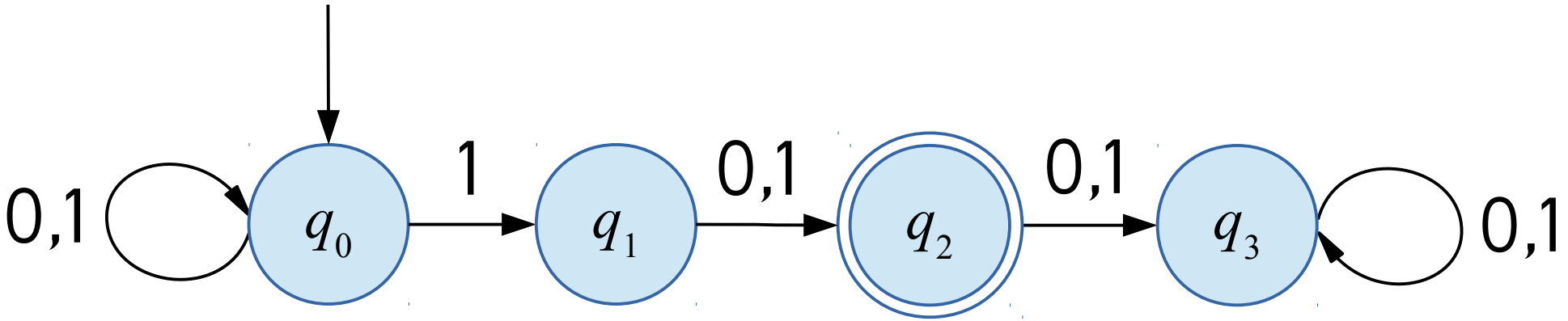


Answer: All strings ending with 1

Another NFA



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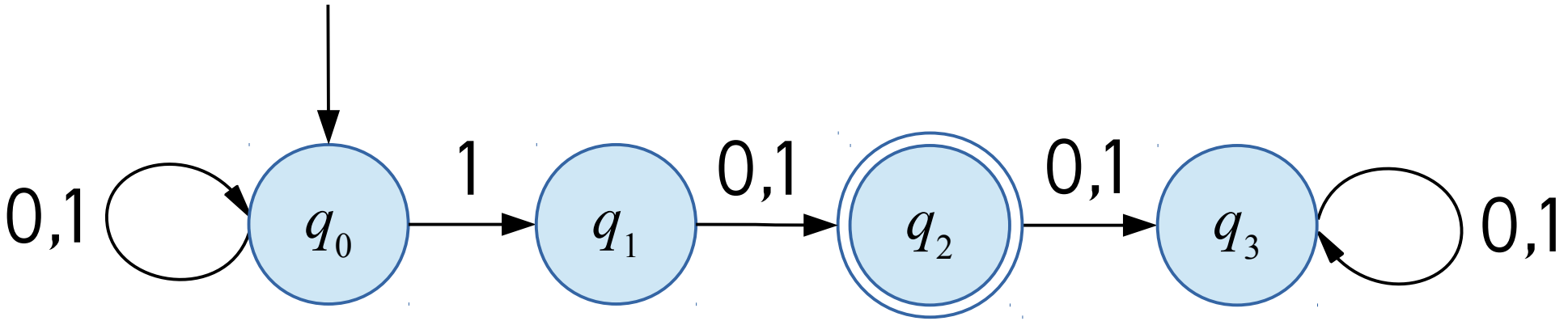

Answer: All strings with 1 in the penultimate place

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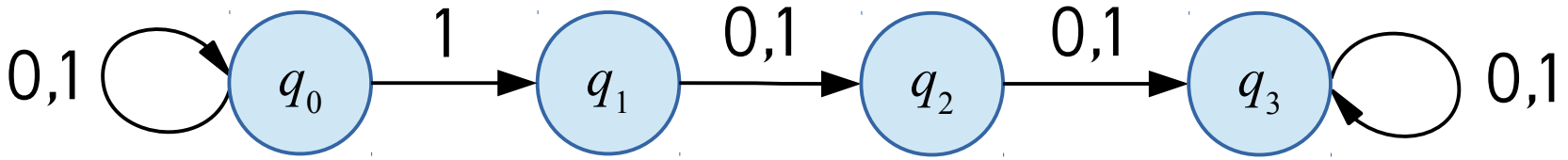

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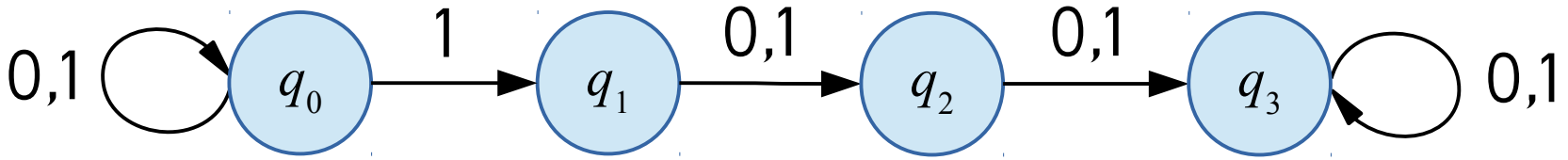

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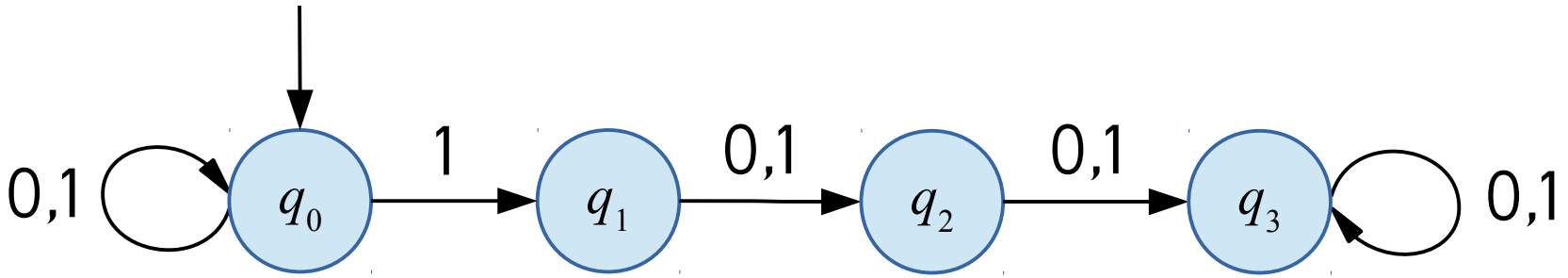

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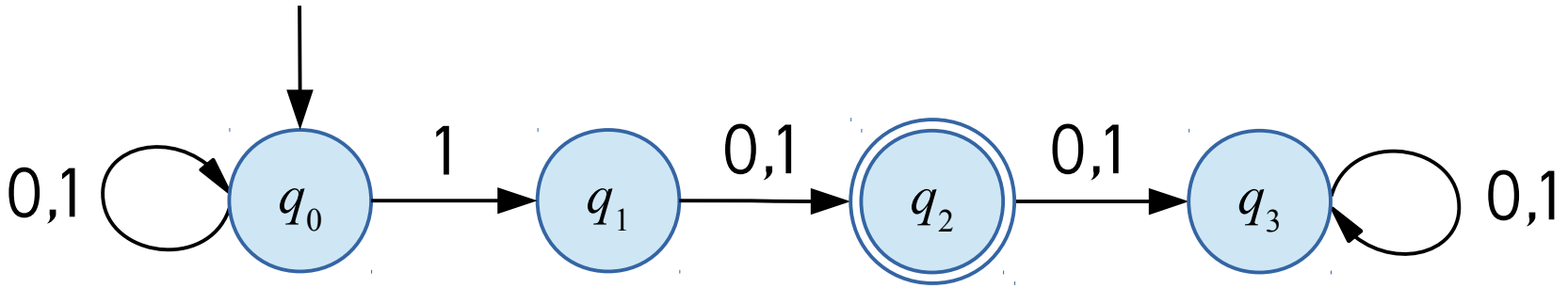

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  - A convenient shortcut for our "hell/black-hole" state

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  - ... i.e. they accept exactly the same language
  - Exponential blowup: if the NFA has n states, the DFA can require up to 2^n states

Thought for the Day #1

Find an NFA with n states that can't be simulated by a DFA with less than 2^n states

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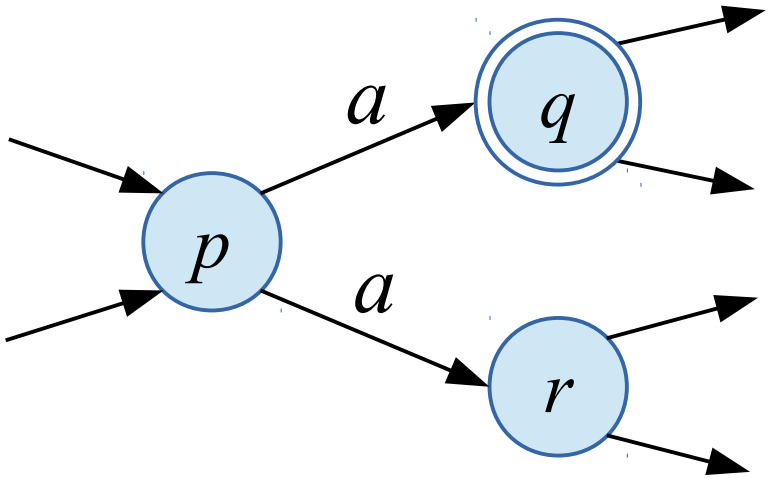

NFA
(fragment)

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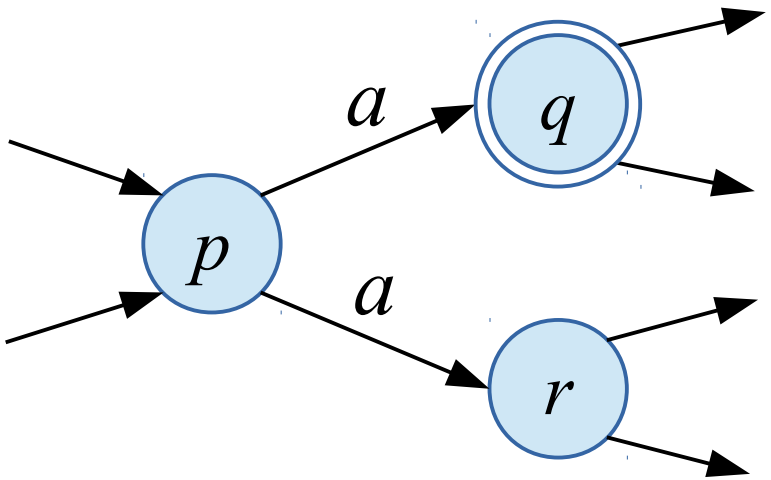

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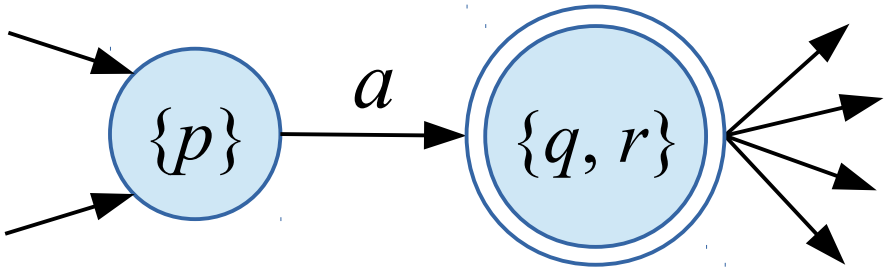
DFA (fragment)



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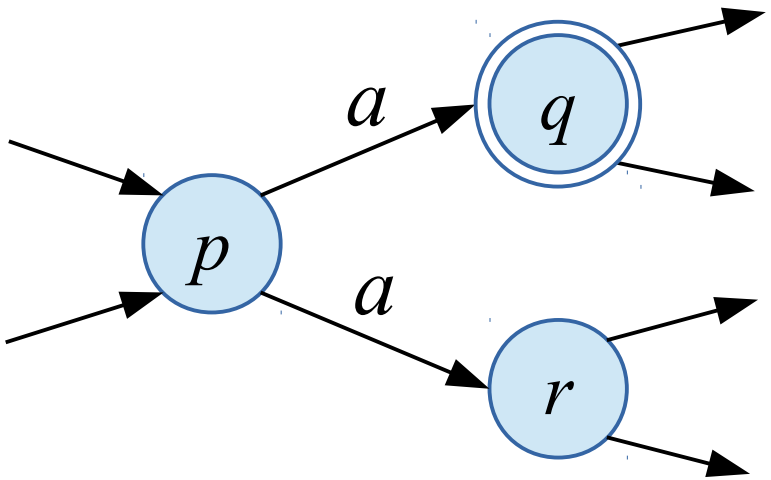
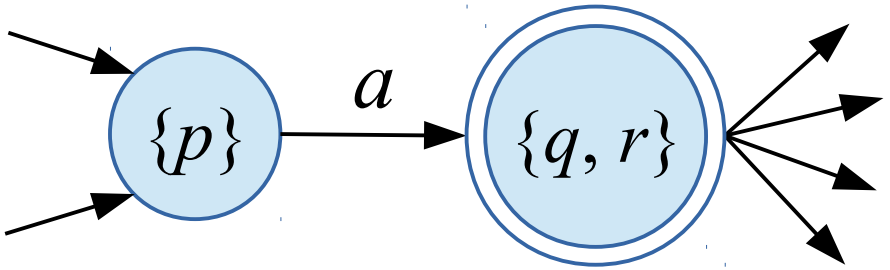


NFA
(fragment)

"Subset construction"

DFA (fragment)



(This is merely illustrative – formal construction on following slides)

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Transition Function		
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Accepting States	F	$F' = \{S \in 2^Q \mid S \cap F \neq \emptyset\}$

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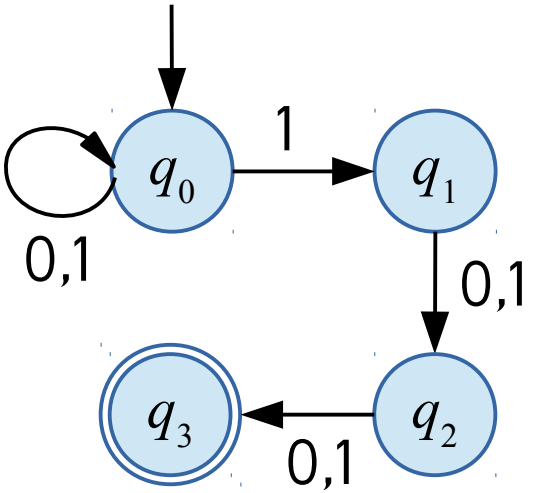

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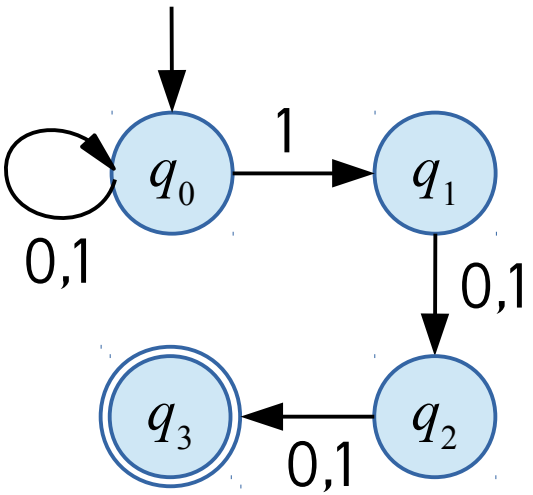

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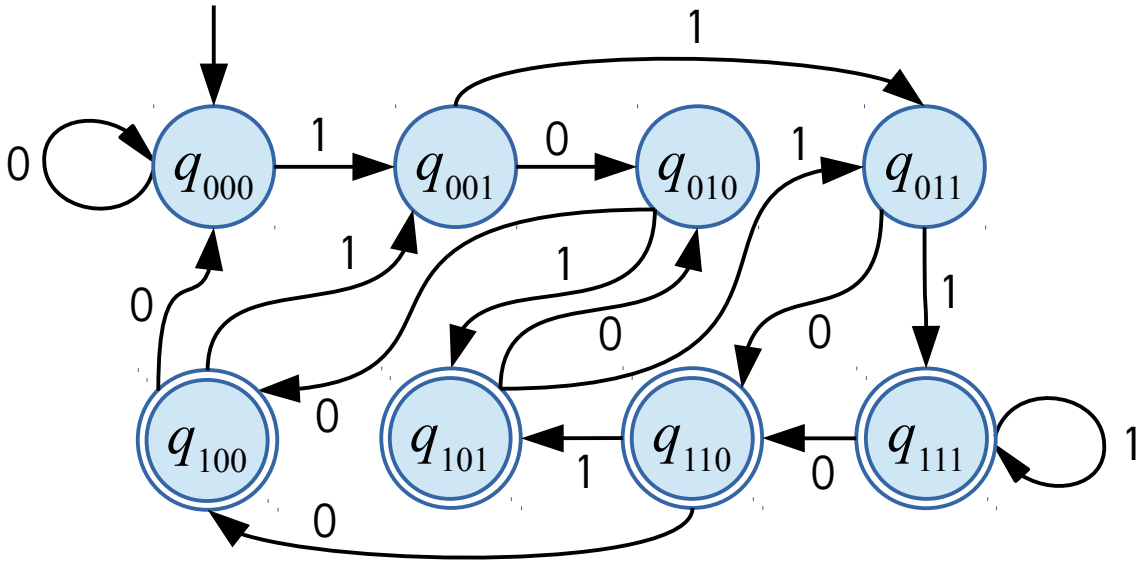

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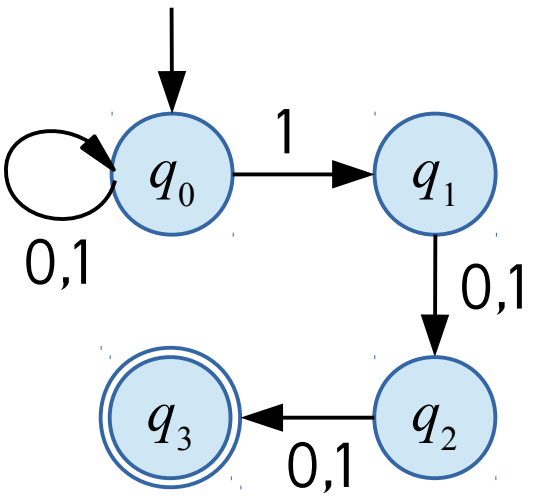

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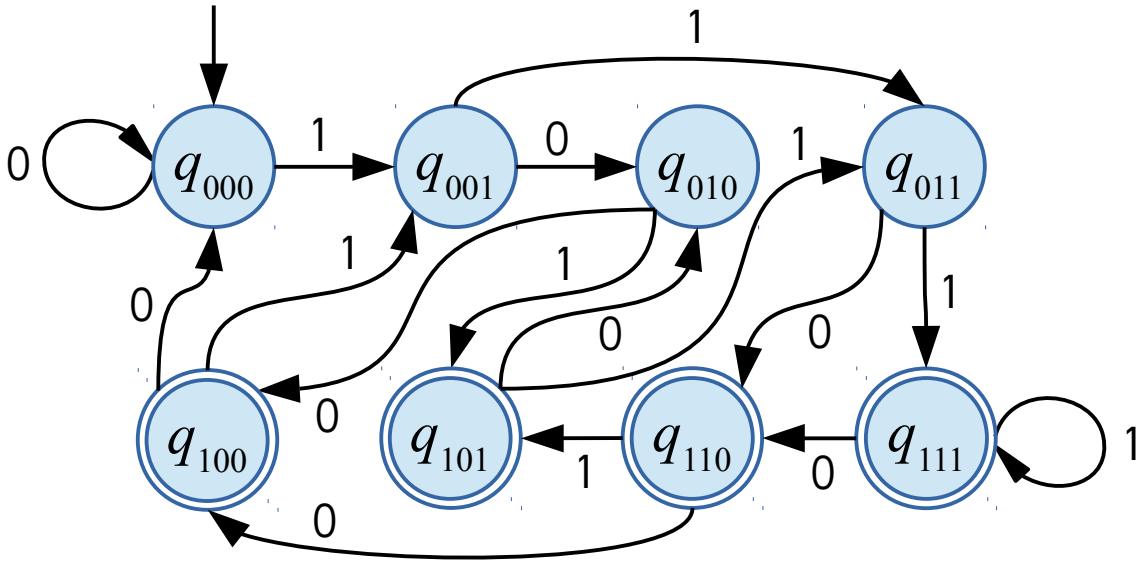

Equivalent minimal DFA

- It's easier to directly convert regular expressions ("wildcard search" on steroids) to NFAs

Finite automata struggle to count

- Any FA (deterministic or not) that recognizes the language $\{1^c\}$ (the single string of c 1's) must have more than c states
  - The parent alphabet is irrelevant (but must of course contain 1)

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  - Let $t = j - i$

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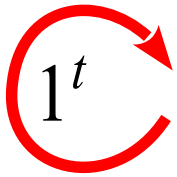

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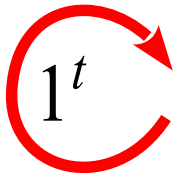

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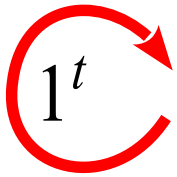

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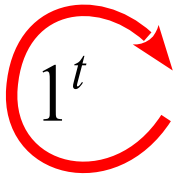

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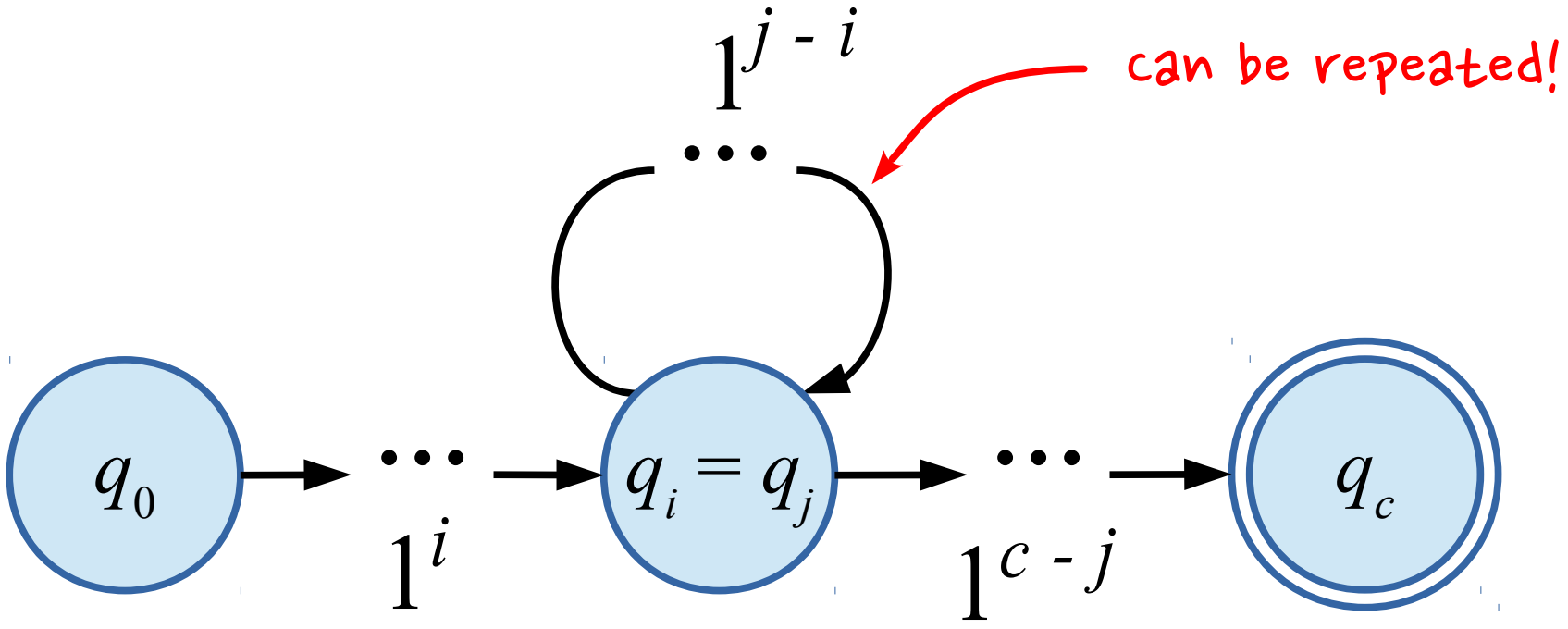
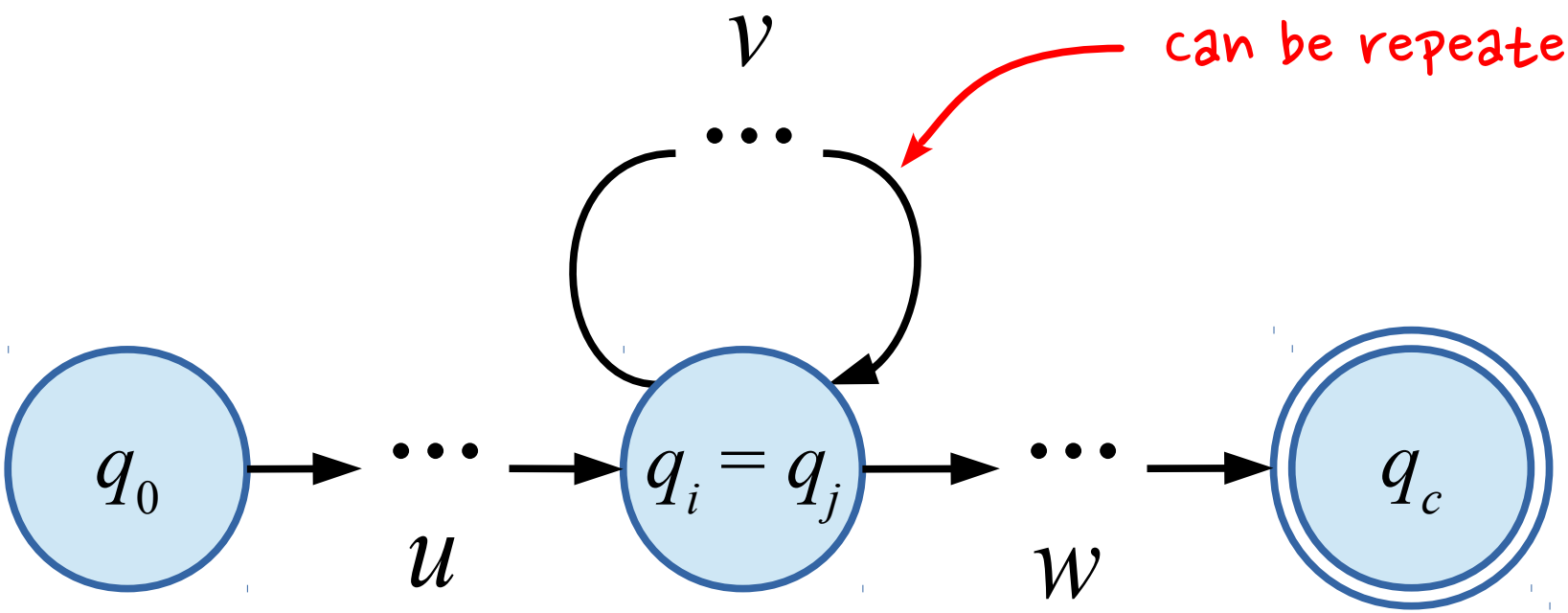


Generalizing...



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- If M is an FA with k states
- ... and M accepts some string x with $|x| \geq k$,
- ... then there exist strings u, v and w such that
  - $x = uvw$,
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Exercise: write out the proof formally!

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  - FA's don't have unlimited “memory”, so they can't store the number of 0's they have seen

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