Proofs and Cardinality

CS 2800: Discrete Structures, Fall 2014

Sid Chaudhuri

$$1 + 2 + 3 + 4 + 5 + \ldots = ???$$

$$1 + 2 + 3 + 4 + 5 + \ldots = -\frac{1}{12}$$

$$1 + 2 + 3 + 4 + 5 + \ldots = -\frac{1}{12}$$

Pull the other one!

http://youtu.be/w-I6XTVZXww

Also see:

http://en.wikipedia.org/wiki/1_%2B_2_%2B_3_%2B_4_%2B_%E2%8B%AF

http://terrytao.wordpress.com/2010/04/10/the-euler-maclaurin-formula-bernoulli-numbers-the-zeta-function-and-real-variable-analytic-continuation/

What are the flaws in this “proof”?

But there's more to this result than meets the eye...

Another way of finding the constant is as follows — 41:
Let us take the series $1+2+3+4+5+\&c$. Let C be its con-
-stant. Then $C = 1+2+3+4+\&c$

$$\therefore 4C = \quad 4 \quad +8 \quad +\&c$$

$$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$$

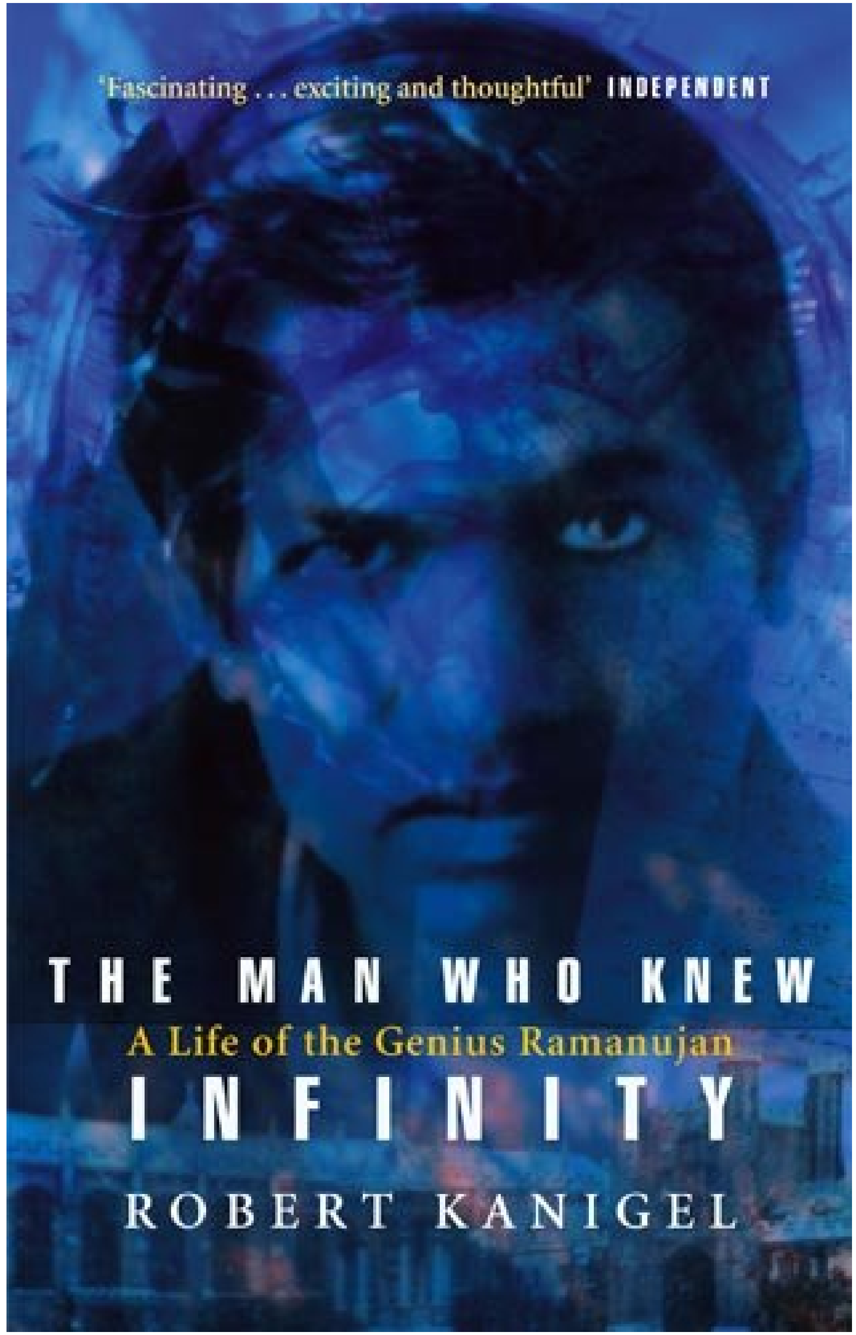
$$\therefore C = -\frac{1}{12}.$$

Srinivasa Ramanujan, Notebook 1, Chapter 8

“Dear Sir,

I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's *Infinite Series* and not fall into the pitfalls of divergent series. ... I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \cdots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter.”

Srinivasa Ramanujan to G. H. Hardy, 27 Feb 1913


'Fascinating . . . exciting and thoughtful' INDEPENDENT
THE MAN WHO KNEW
A Life of the Genius Ramanujan
INFINITY
ROBERT KANIGEL

The Riemann Zeta Function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots$$

for complex numbers s with real part > 1

The Riemann Zeta Function

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For $\mathrm{Re}(s) \leq 1$, the series diverges, but $\zeta(s)$ can be defined by a process called *analytic continuation*.

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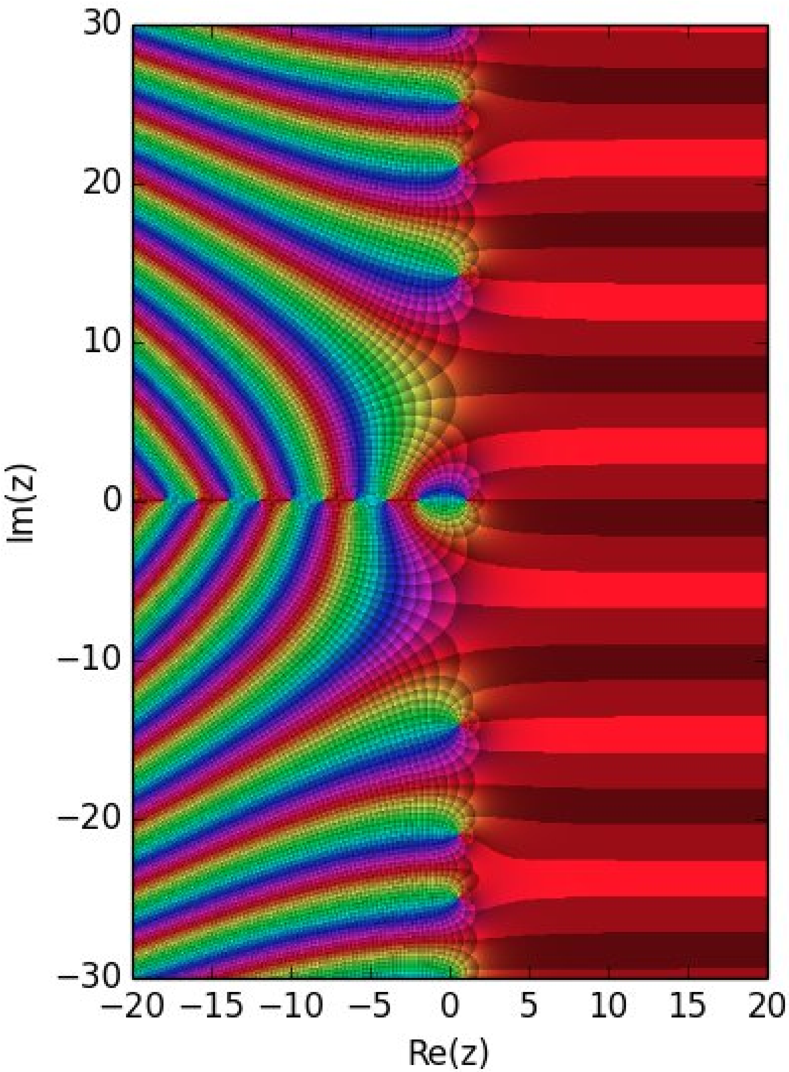
For $\mathrm{Re}(s) \leq 1$, the series diverges, but $\zeta(s)$ can be defined by a process called *analytic continuation*.

This also gives $\zeta(-1) = -1/12$!

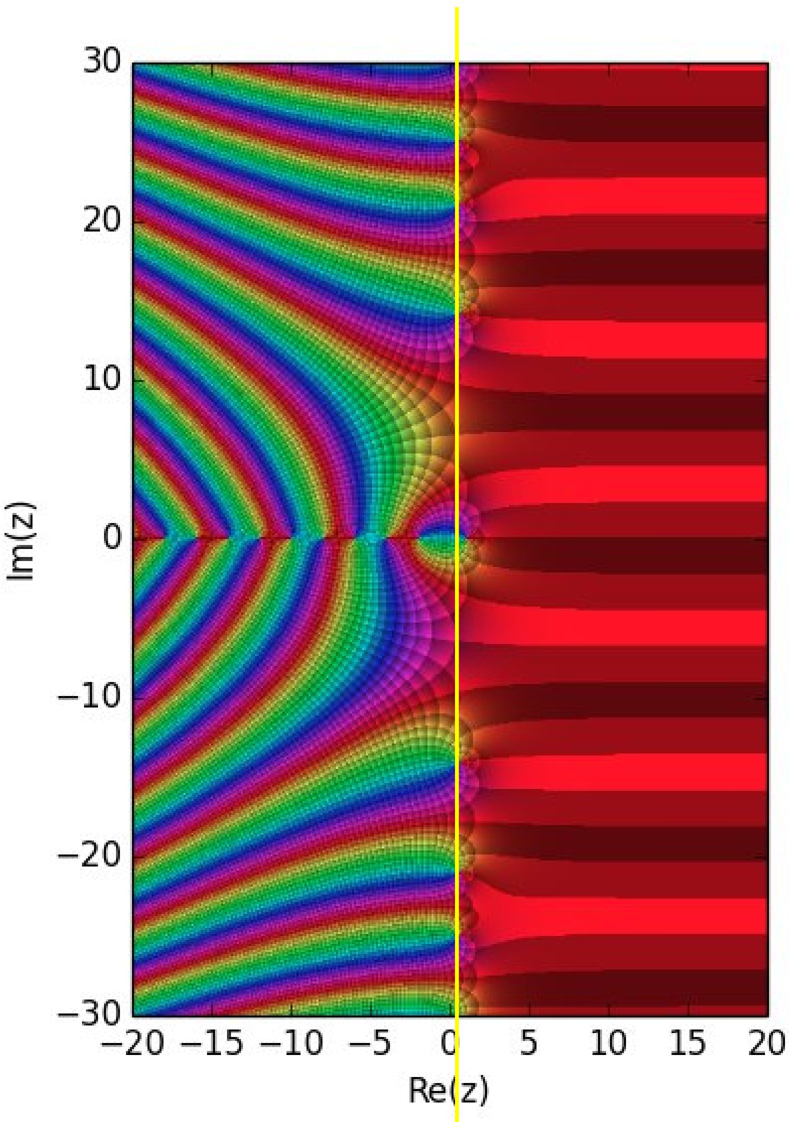
The Riemann Hypothesis



All the “non-trivial” zeros of the Riemann zeta function $\zeta(s)$ have real part ½

Empetrisor@Wikipedia

The Riemann Hypothesis



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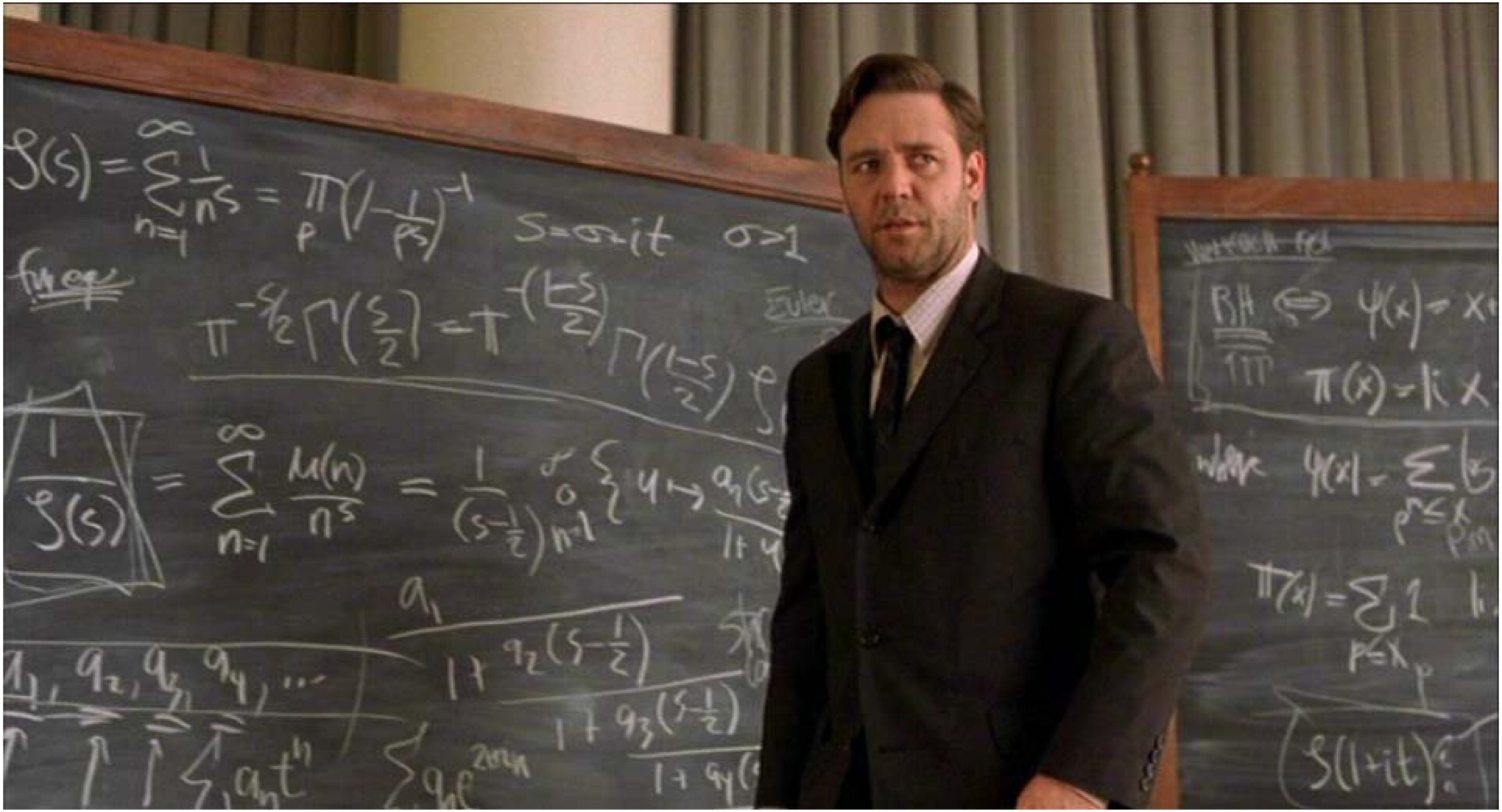
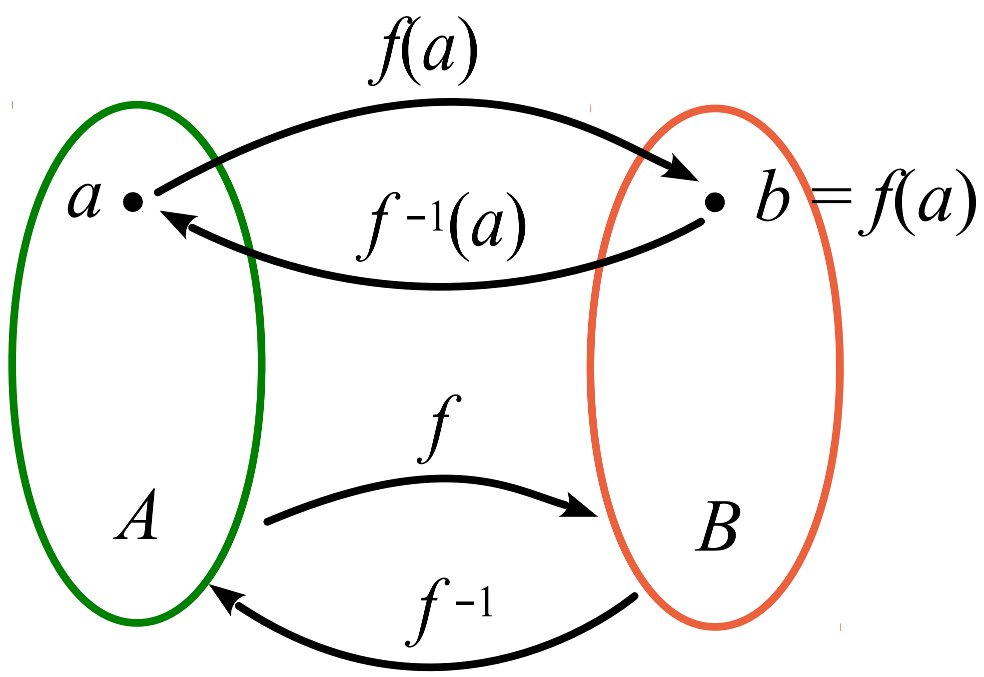


… And so we see that if the zeroes of the Riemann zeta function correspond to singularities in the space time then conventional number theory breaks down in the face of relativistic exploration… Sometimes, our expectations are betrayed by the numbers… And variables are impossible to assign any rational value…


WANTED
DEAD OR ALIVE
REWARD $ 1,000,000

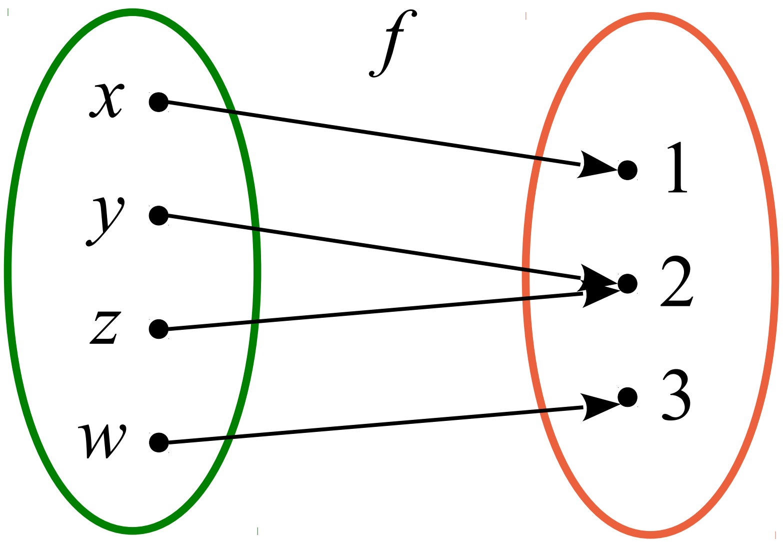
Inverse of a function

- The inverse of a bijective function $f: A \rightarrow B$ is the unique function $f^{-1}: B \rightarrow A$ such that for any $a \in A, f^{-1}(f(a)) = a$ and for any $b \in B, f(f^{-1}(b)) = b$
- A function is bijective iff it has an inverse function



Inverse of a function

- If f is not a bijection, it cannot have an inverse function



Onto, not one-to-one

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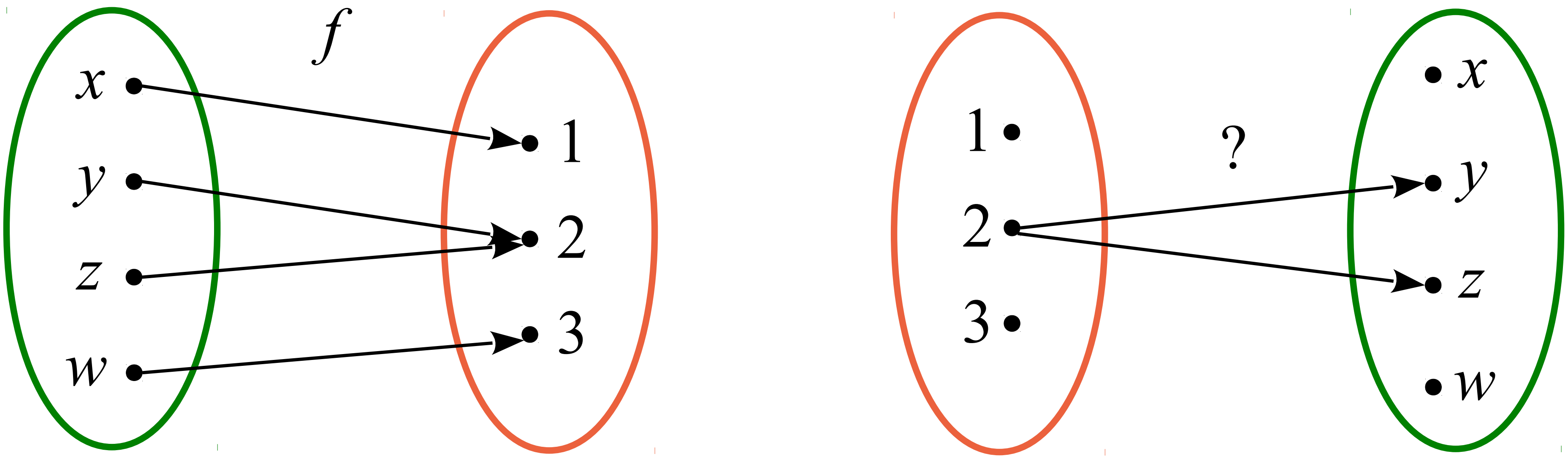

Onto, not one-to-one

$f^{-1}(2) = ?$

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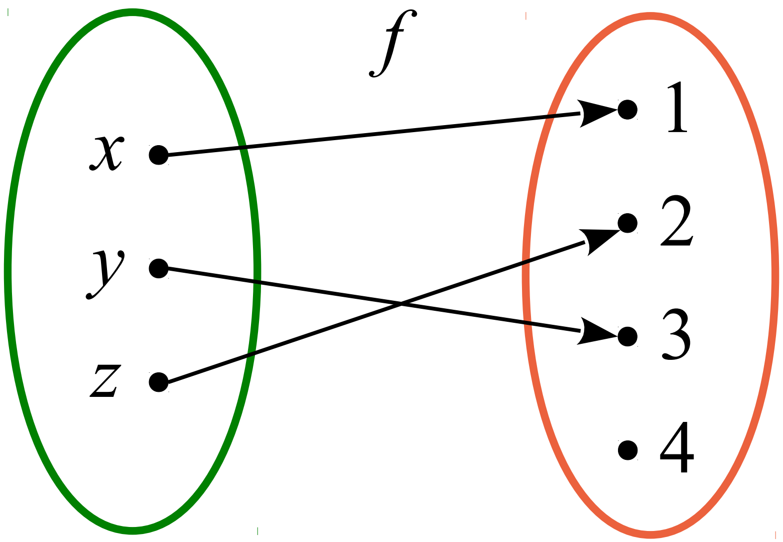

One-to-one, not onto

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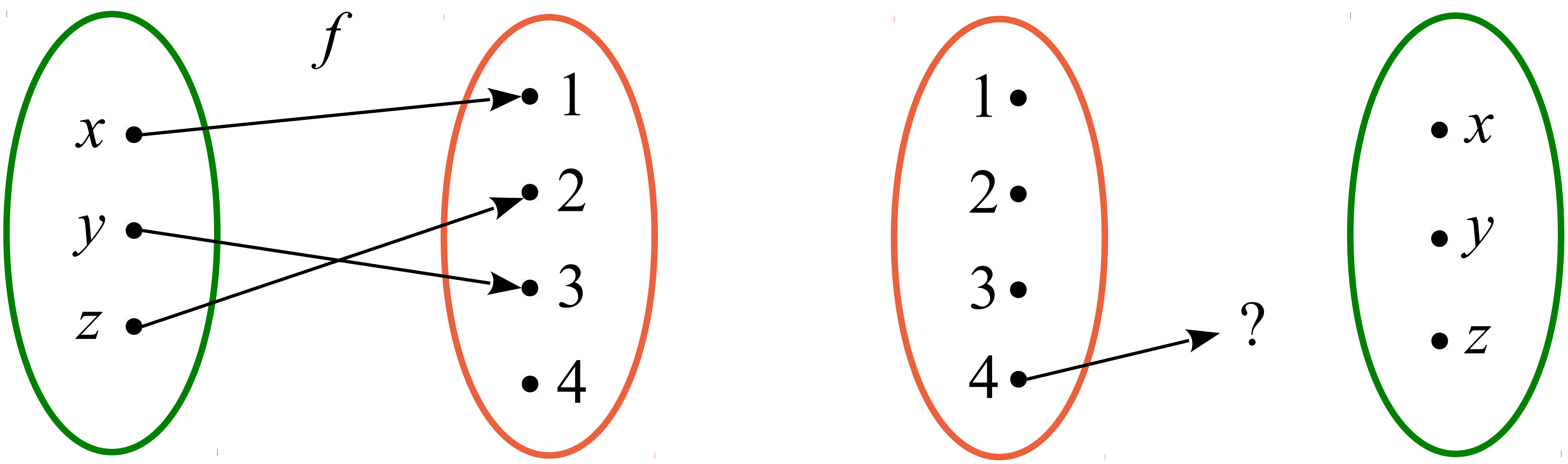
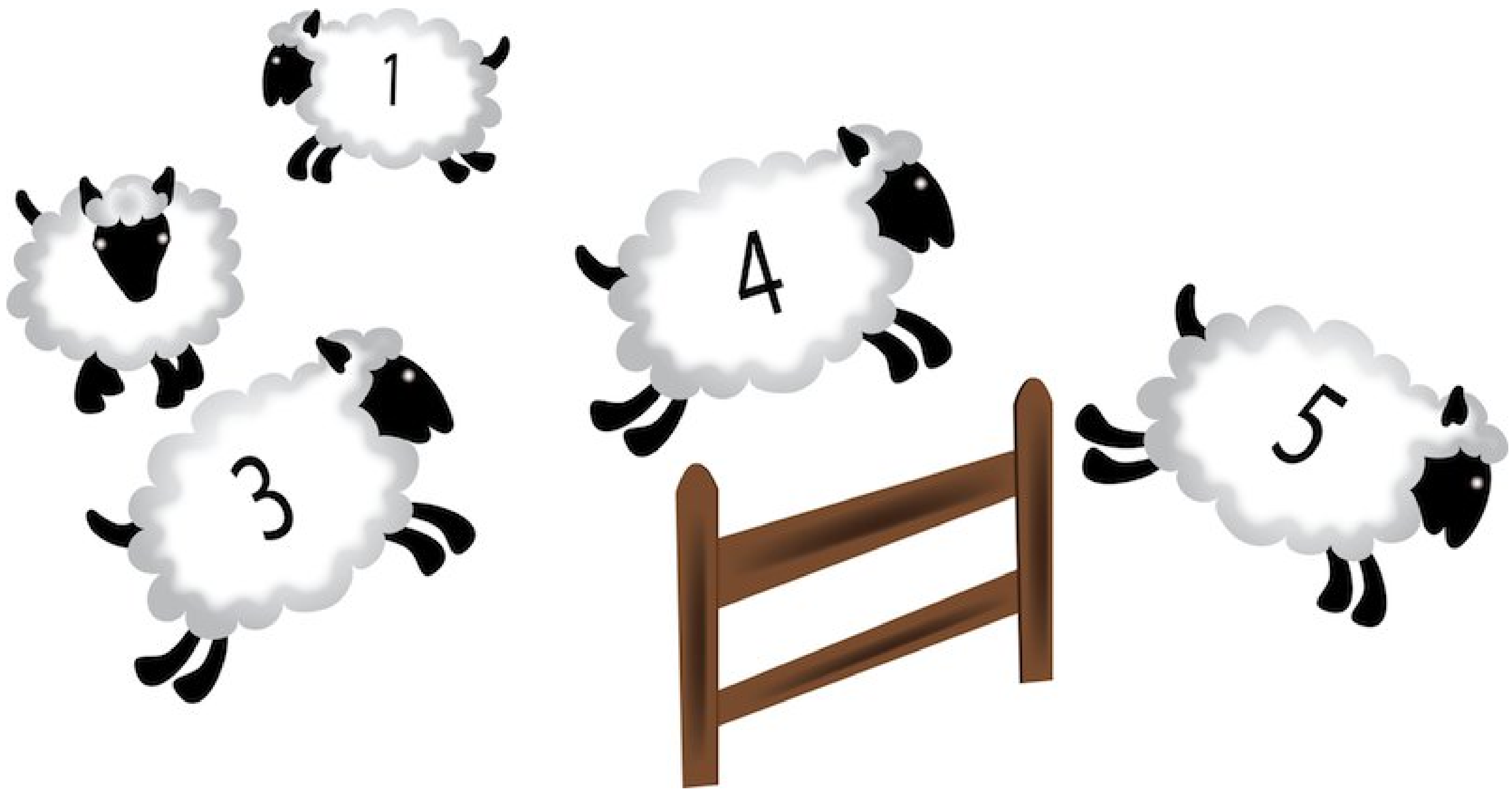


One-to-one, not onto

$f^{-1}(4) = ?$

How can we count elements in a set?



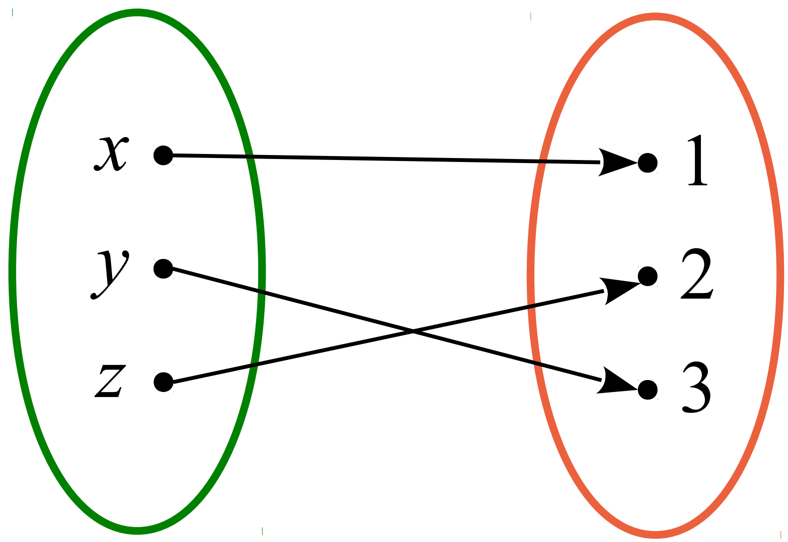
How can we count elements in a set?

- Easy for finite sets – just count the elements!
- Does it even make sense to ask about the number of elements in an infinite set?
- Is it meaningful to say one infinite set is larger than another?
  - Are the natural numbers larger than
    - the even numbers?
    - the rational numbers?
    - the real numbers?

Cardinality and Bijections

- If A and B are finite sets, clearly they have the same number of elements iff there is a bijection between them

$$\text{e.g. } |\{x, y, z\}| = |\{1, 2, 3\}| = 3$$



Following Ernie Croot's slides

Cardinality and Bijections

- **Definition:** Sets A and B have the same ***cardinality*** iff there is a bijection between them

Cardinality and Bijections

- **Definition:** Sets A and B have the same ***cardinality*** iff there is a bijection between them
  - For finite sets, cardinality is the number of elements
  - There is a bijection between n-element set A and $\{1, 2, 3, \ldots, n\}$

Cardinality and Bijections

- Natural numbers and even numbers have the same cardinality

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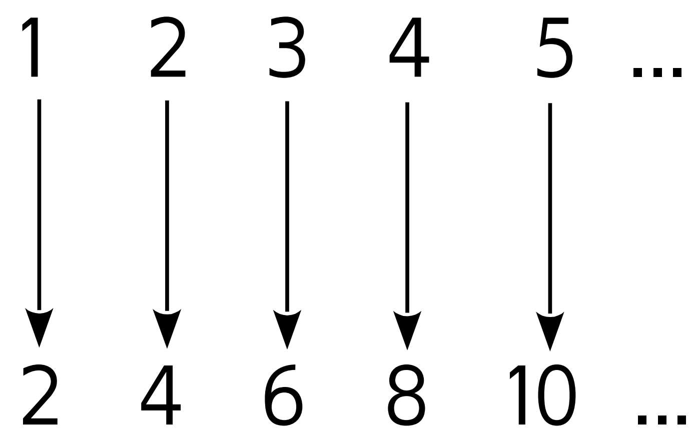

Cardinality and Bijections

- Natural numbers and even numbers have the same cardinality

1	2	3	4	5	...
↓	↓	↓	↓	↓	
2	4	6	8	10	...

Sets having the same cardinality as the natural numbers (or some subset of the natural numbers) are called <u>countable sets</u>

Following Ernie Croot's slides

Cardinality and Bijections

- Natural numbers and rational numbers have the same cardinality!

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Illustrating proof only for positive rationals here, can be easily extended to all rationals

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & & \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & & \\
 & & & \ddots & \\
\frac{4}{1} & & & & \\
\vdots & & & &
\end{array}
$$

Cardinality and Bijections

- Natural numbers and rational numbers have the same cardinality!

Illustrating proof only for positive rationals here, can be easily extended to all rationals

① $\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\cdots$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$		
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$		
$\frac{4}{1}$			$\ddots$	
$\vdots$				

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$$
\begin{array}{ccccc}
① \frac{1}{1} & \rightarrow ② \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & & \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & & \\
\frac{4}{1} & & & \ddots & \\
\vdots & & & &
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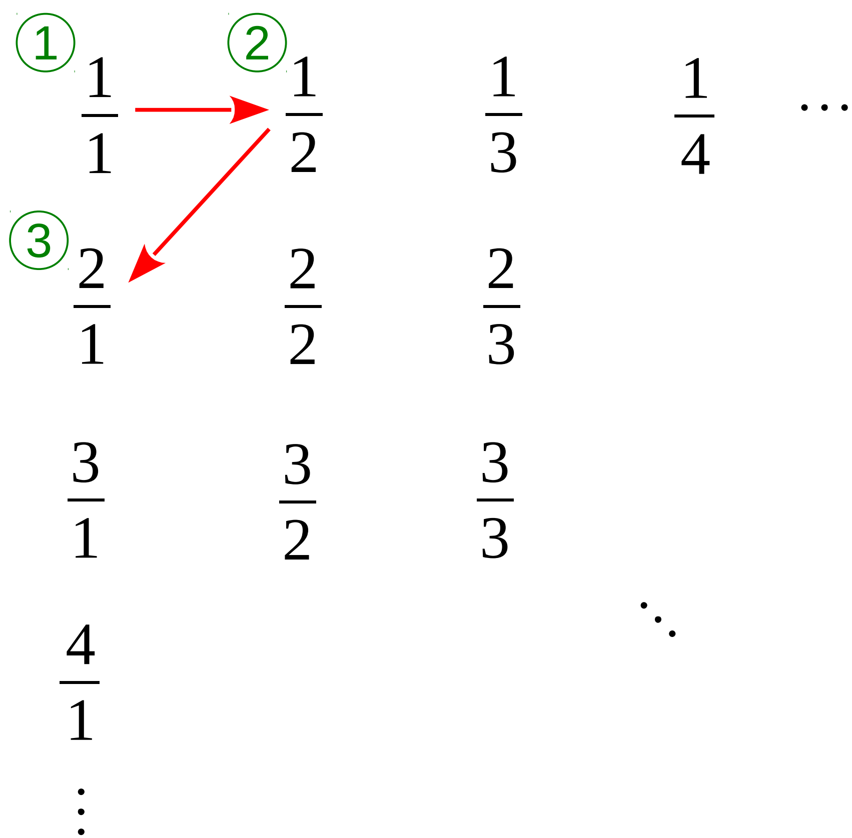

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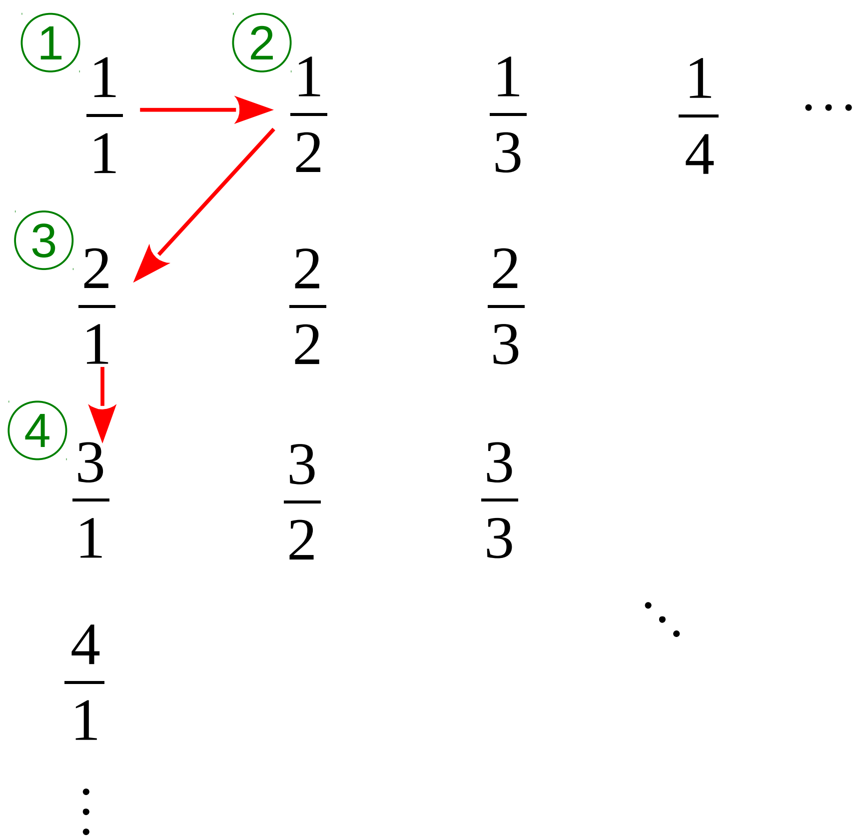

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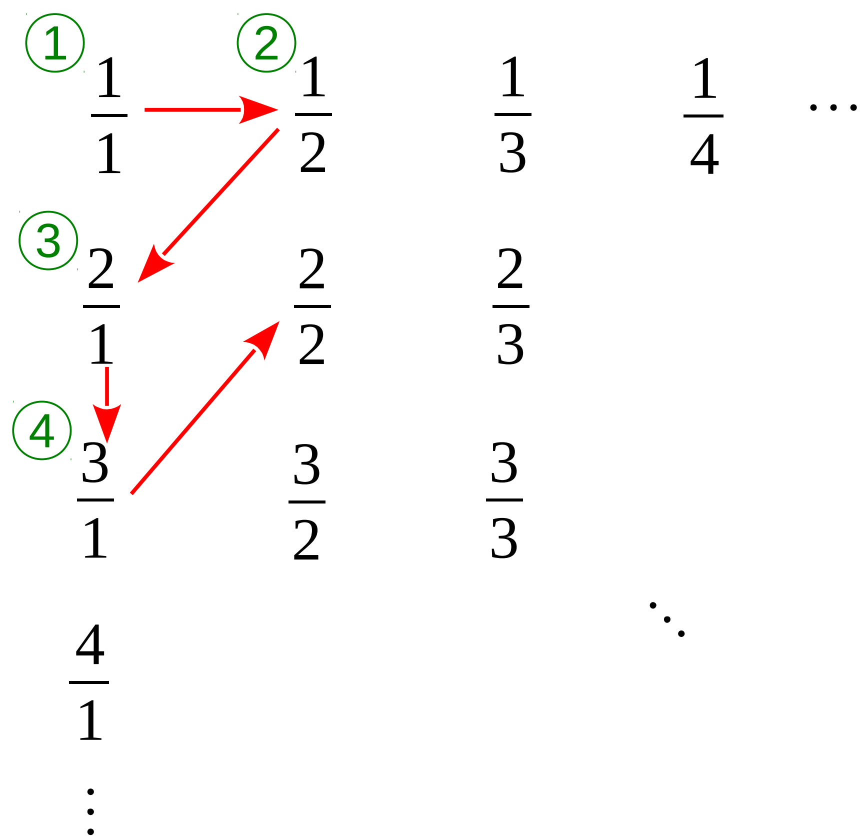

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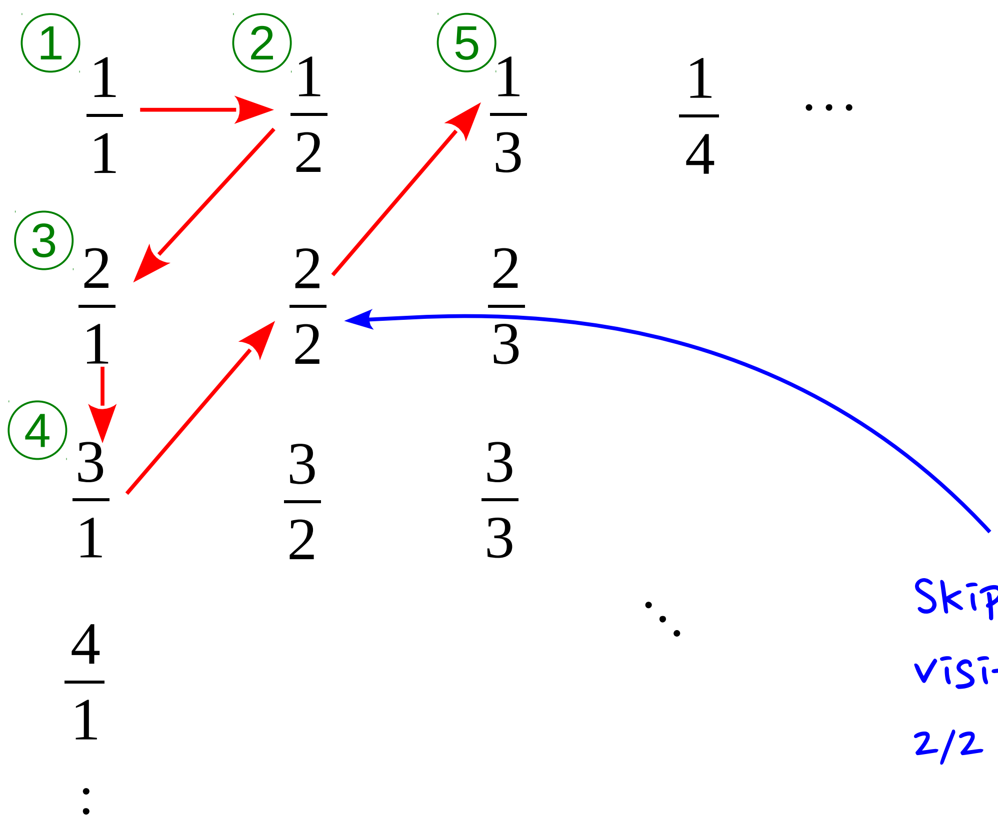

Skip previously visited rationals: 2/2 = 1/1

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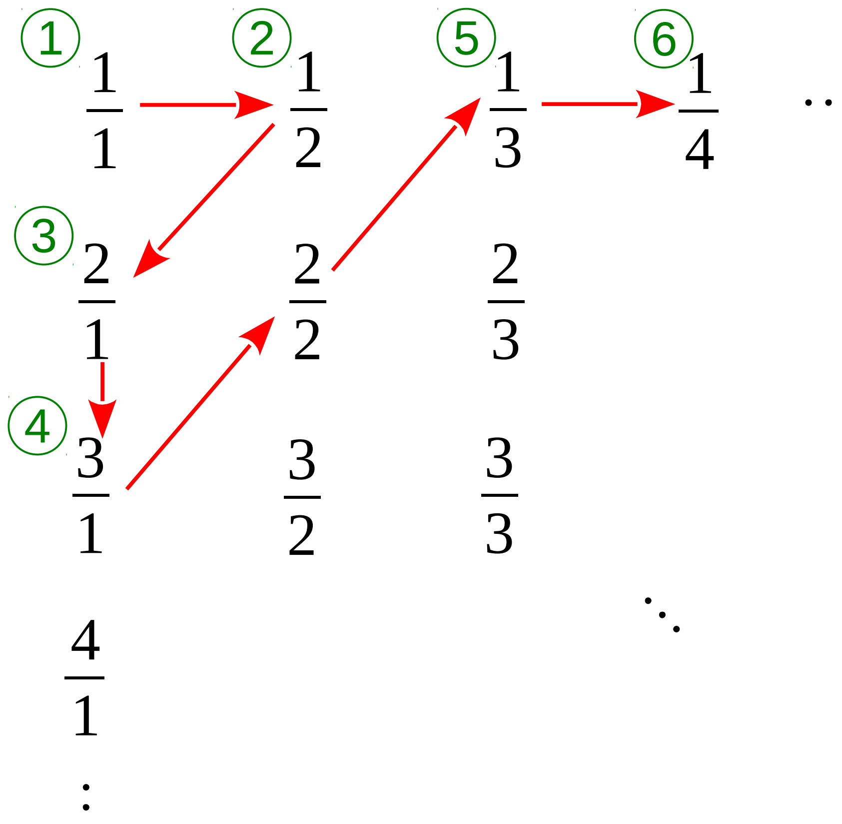

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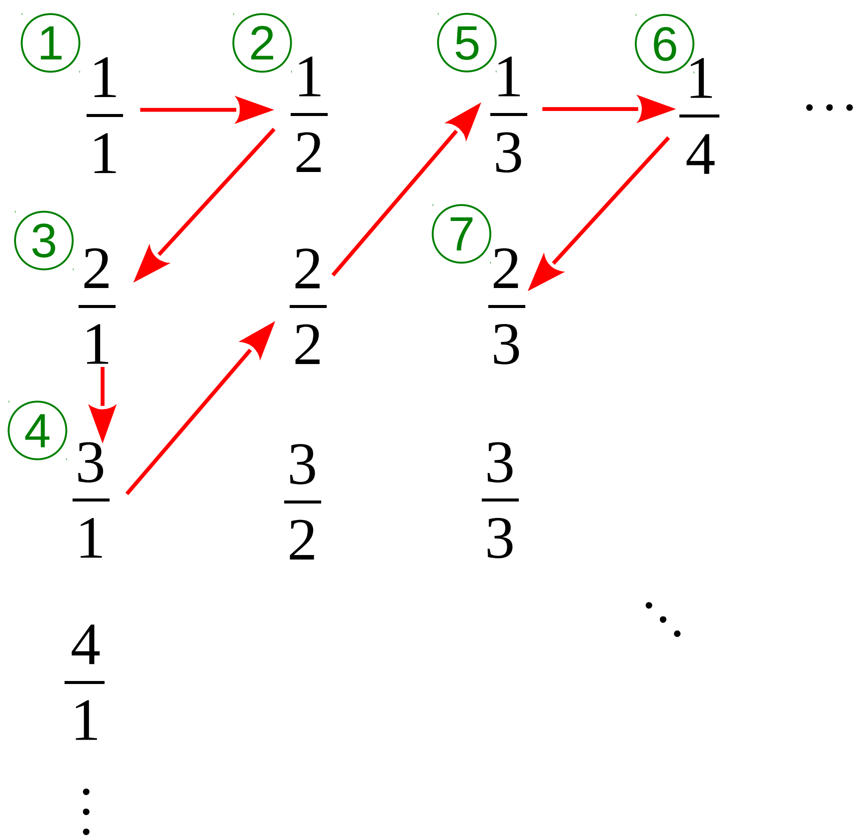

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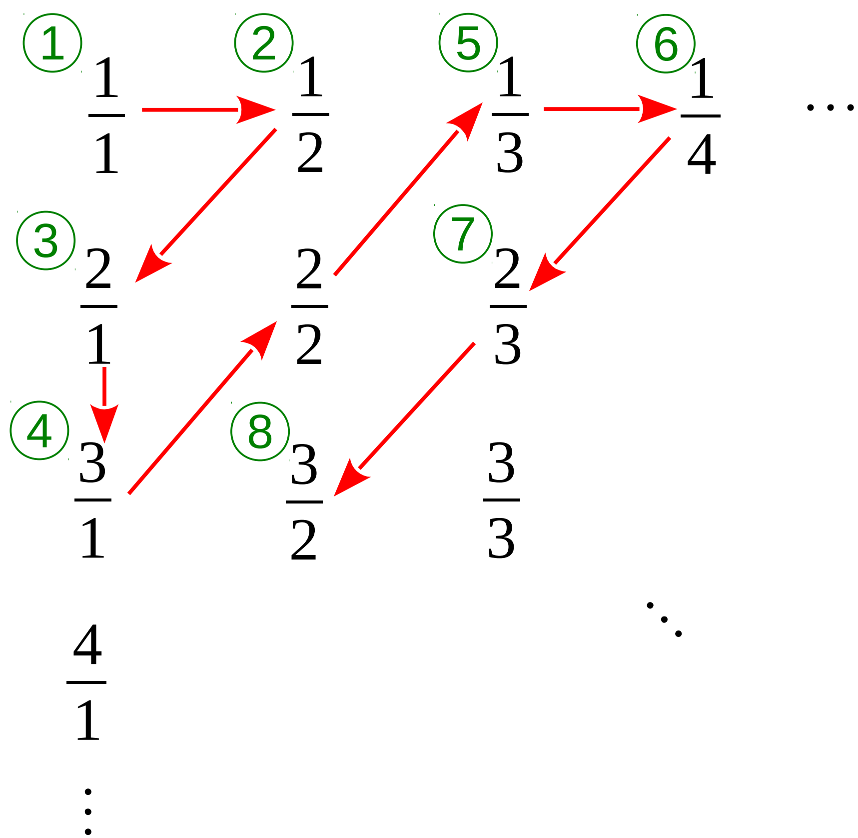

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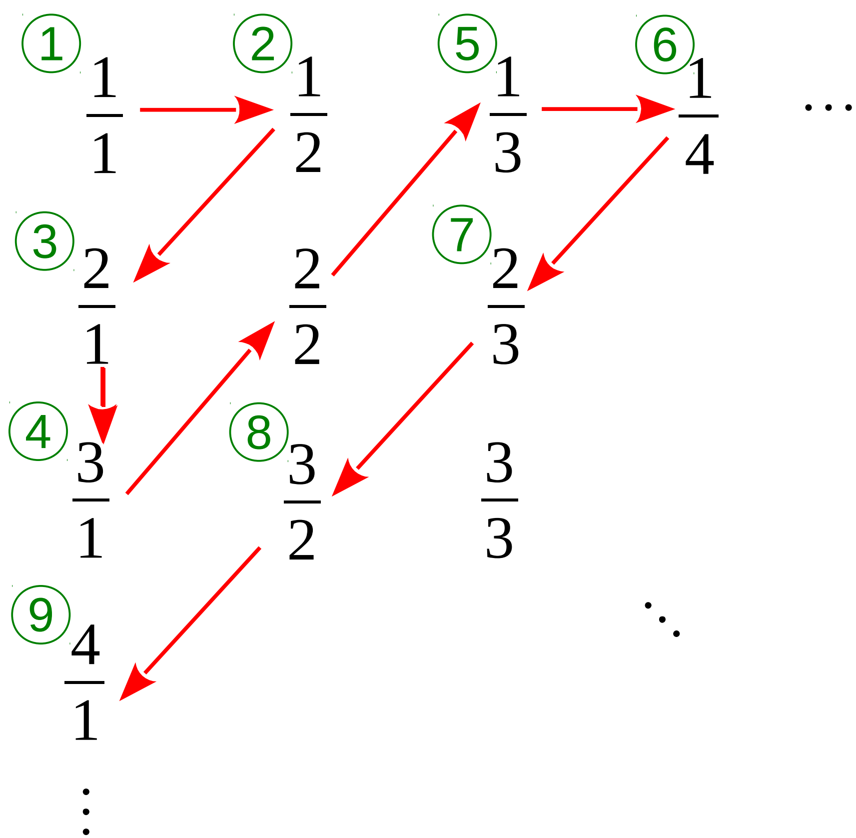

Cardinality and Bijections

- The natural numbers and real numbers *do not* have the same cardinality

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x_1	0 . 0 0 0 0 0 0 0 0 0 …
x_2	0 . 1 0 3 0 4 0 5 0 1 …
x_3	0 . 9 8 7 6 5 4 3 2 1 …
x_4	0 . 0 1 2 1 2 1 2 1 2 …
x_5	⋮

Cardinality and Bijections

- The natural numbers and real numbers *do not* have the same cardinality

$$
\begin{array}{c|l}
x_1 & 0\,.\,\mathbf{0}\,0\,0\,0\,0\,0\,0\,0\,0\,\ldots \\
x_2 & 0\,.\,1\,\mathbf{0}\,3\,0\,4\,0\,5\,0\,1\,\ldots \\
x_3 & 0\,.\,9\,8\,\mathbf{7}\,6\,5\,4\,3\,2\,1\,\ldots \\
x_4 & 0\,.\,0\,1\,2\,\mathbf{1}\,2\,1\,2\,1\,2\,\ldots \\
x_5 & \vdots
\end{array}
$$

Cardinality and Bijections

- The natural numbers and real numbers *do not* have the same cardinality

x_1	$0\,.\,\mathbf{0}\,0\,0\,0\,0\,0\,0\,0\,0\ldots$
x_2	$0\,.\,1\,\mathbf{0}\,3\,0\,4\,0\,5\,0\,1\ldots$
x_3	$0\,.\,9\,8\,\mathbf{7}\,6\,5\,4\,3\,2\,1\ldots$
x_4	$0\,.\,0\,1\,2\,\mathbf{1}\,2\,1\,2\,1\,2\ldots$
x_5	$\vdots$

Consider the number

$$y = 0\,.\,b_1\,b_2\,b_3\ldots$$

$$b_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ decimal place of } x_i \text{ is zero} \\ 0 & \text{if it is non-zero} \end{cases}$$

Cardinality and Bijections

- The natural numbers and real numbers *do not* have the same cardinality

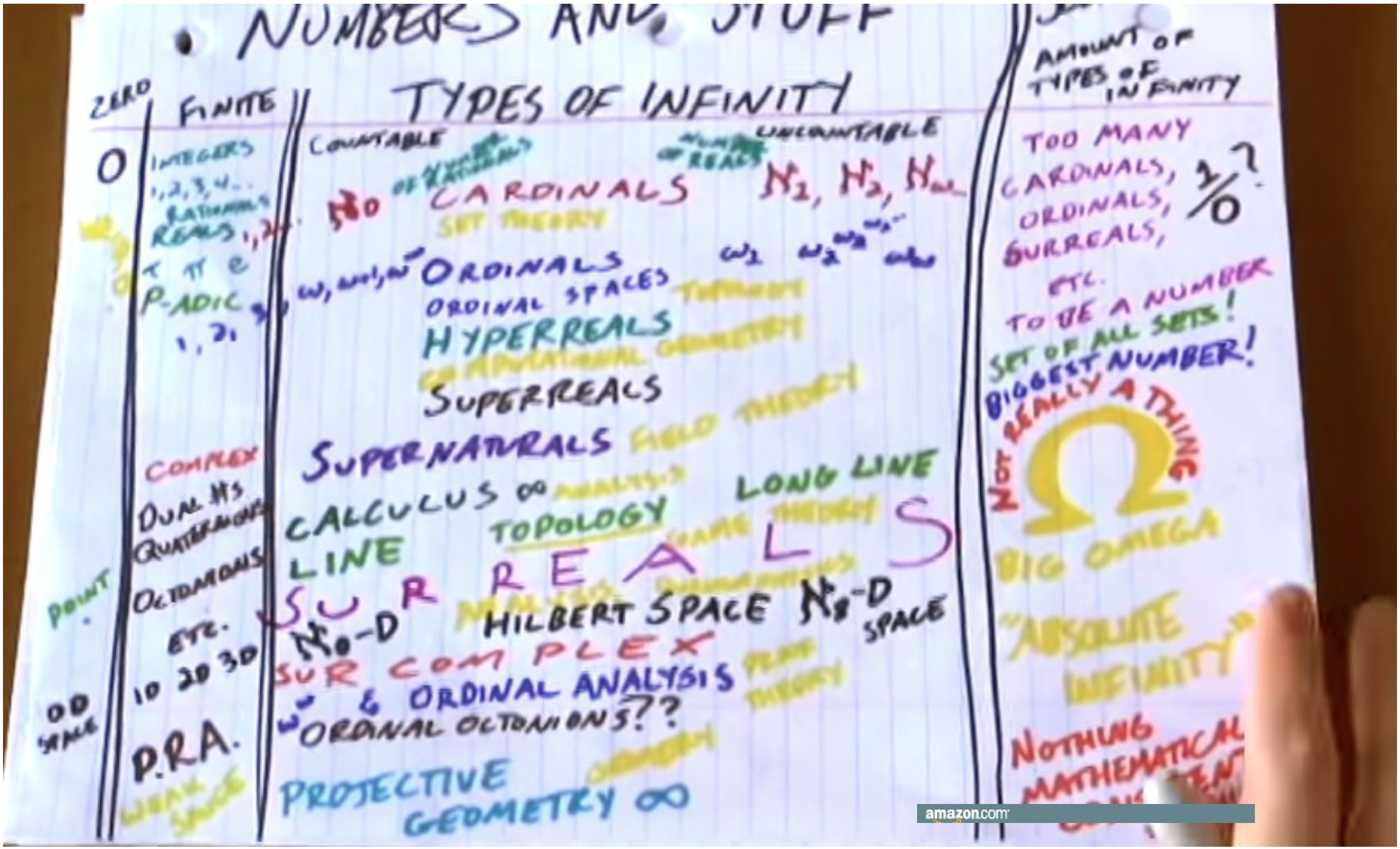
x_1	0	.	**0**	0	0	0	0	0	0	0	0	…
x_2	0	.	1	**0**	3	0	4	0	5	0	1	…
x_3	0	.	9	8	**7**	6	5	4	3	2	1	…
x_4	0	.	0	1	2	**1**	2	1	2	1	2	…
x_5	⋮											

Consider the number
$y = 0 . b_1 b_2 b_3 \ldots$

$$b_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ decimal place of } x_i \text{ is zero} \\ 0 & \text{if it is non-zero} \end{cases}$$

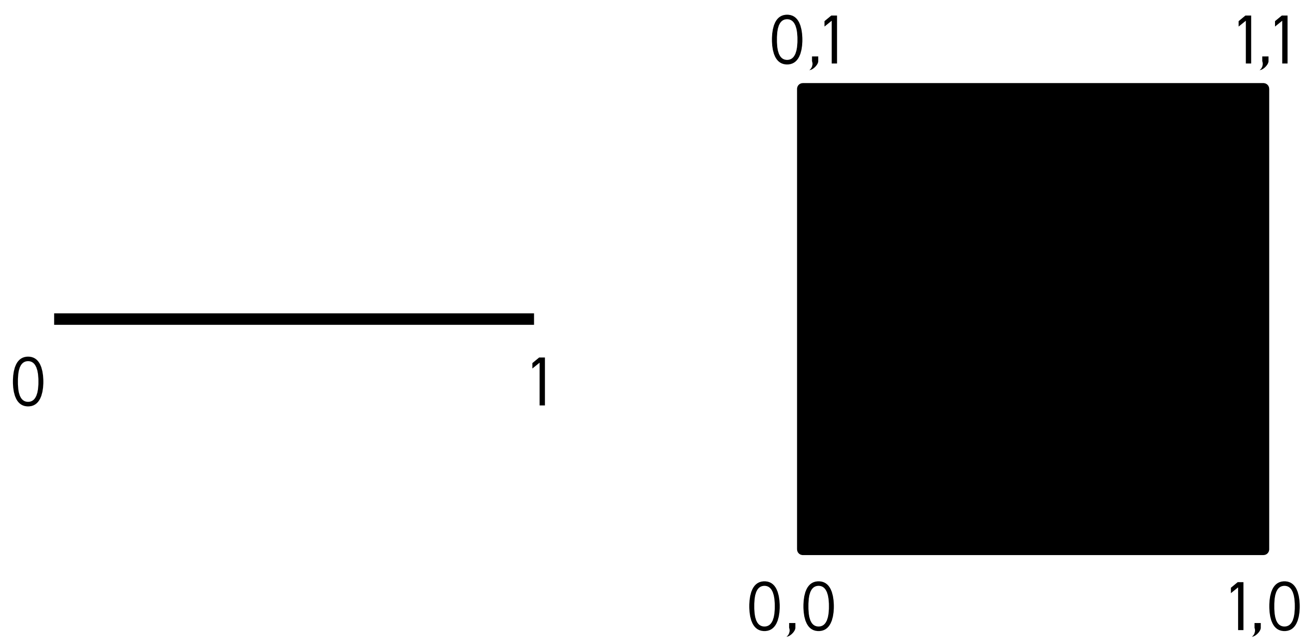
y cannot be equal to any x_i – it differs by one digit from each one!

There are many infinities



Thought for the Day #1

Do the real interval $[0, 1]$ and the unit square $[0, 1] \times [0, 1]$ have the same cardinality?



Comparing Cardinalities

- **Definition:** If there is an injective function from set A to set B, we say $|A| \leq |B|$

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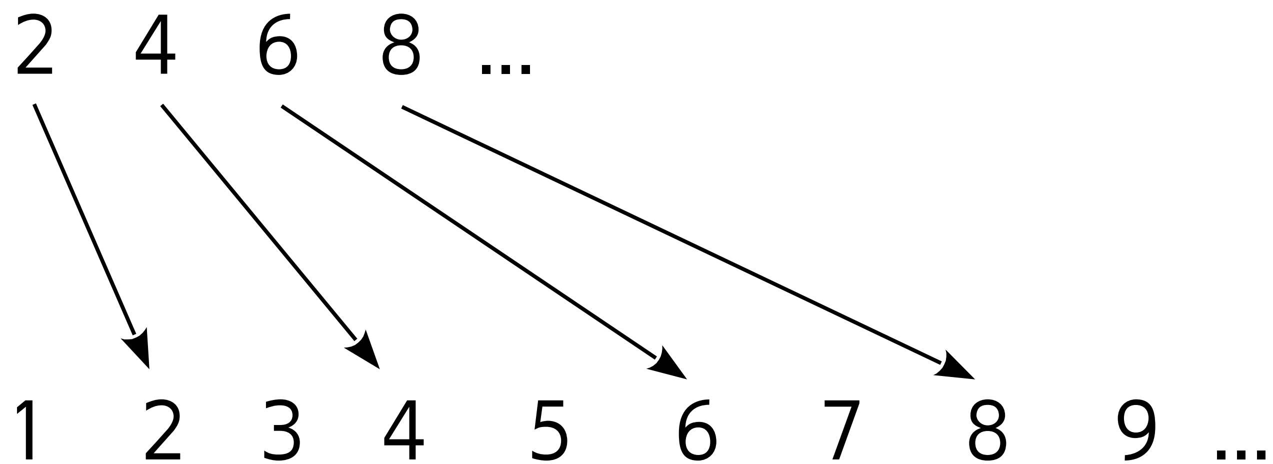

$$|Evens| \leq |\mathbb{N}|$$

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- If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

Comparing Cardinalities

- **Definition:** If there is an injective function from set A to set B, but not from B to A, we say $|A| < |B|$
- If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$
  - *Exercise:* prove there is a bijection from A to B iff there are injective functions from A to B and from B to A