Today

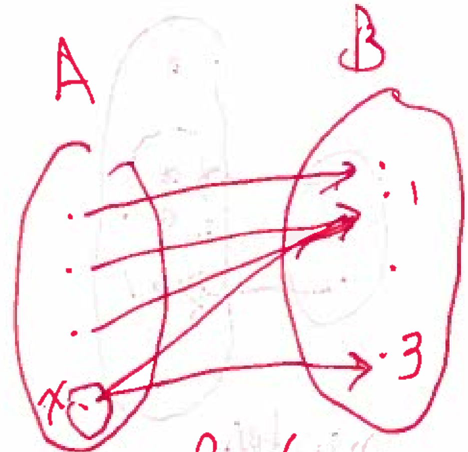
- More on functions
- for all and there exists.

Def: function $f: A \to B$ gives an element of B for every elt of A.

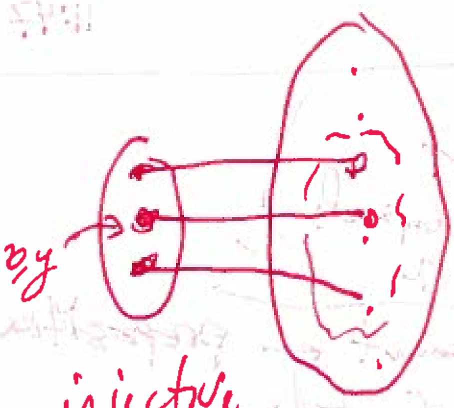
$f: x \mapsto x^2$

f: Dates $\to$ numbers

to compute $f(d)$ look up closing DJI on d.

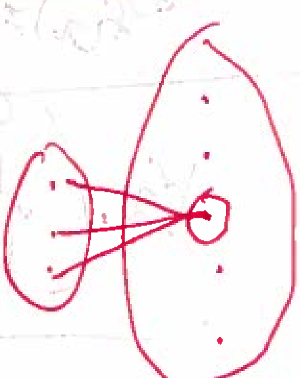


not function



injective

one-to-one



not injective.

Formally:

$f: A \to B$ is injective if for all $x, y \in A$ if $f(x) = f(y)$ then $x = y$.

Definition: $|A| \leq |B|$ if $\exists\, f: A \to B$ that is injective.

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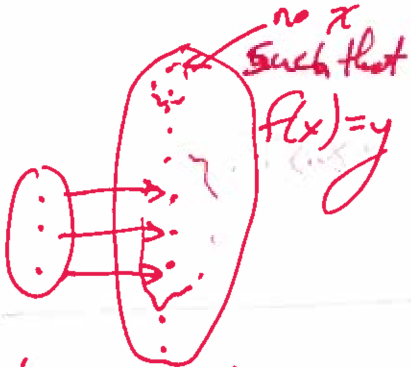
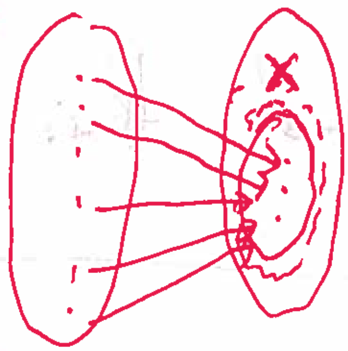


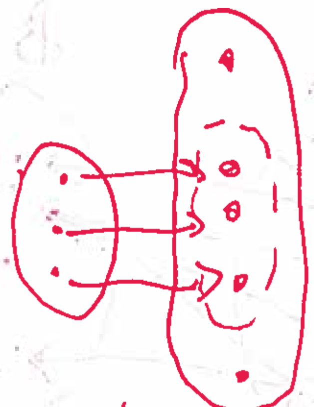
not surjective (elts not "hit" by f)



surjective onto.

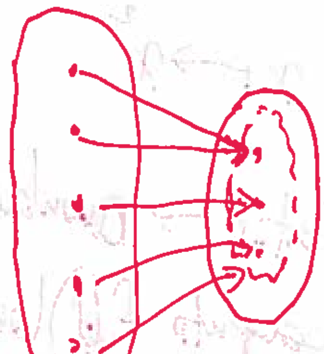
~~Formally~~ **Formally**:

~~For all~~ $f: A \to B$ is surjective (onto) if for every $b \in B$, there is some $a \in A$ such that $f(a) = b$



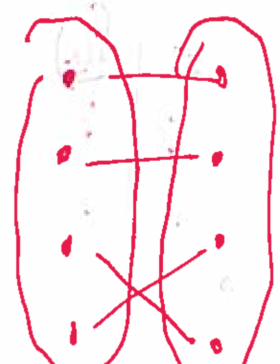
injective

$\forall x, y$, if $f(x) = f(y)$ then $x = y$



surjection

$\forall y \in B, \exists\, x \in A$ such that $f(x) = y$.



bijection! one-to-one onto.

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$\forall$: for all

$\exists$: there exists

$$\underbrace{\forall x, \left(\text{statement that depends on } x \right)}_{\text{statement}}$$

statement $\leftarrow$ propositions. true or false.

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How to prove $\forall x, \underline{P(x)}$
statement depending on x?

$\forall x \in A, P(x)$

- check every possibility.

E.g. Suppose want to show every prime > 2 is odd.

i.e. $\forall x$, if x is prime, $x > 2$ then x is odd

proof:

3. $2 \nmid 3$ ~~thus~~ thus 3 is odd ✓

5. $2 \nmid 5$ and thus 5 is odd ✓

7. $2 \nmid 7$ and thus 7 is odd ✓

takes forever!

Another way: choose arbitrary x, prove for that x. Since x was arbitrary, plug in $3, 5, 7$ ... for x to get individual proof.

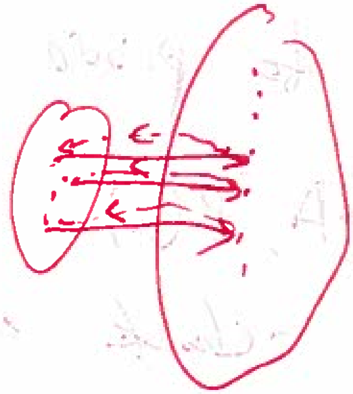
Proof: Let x be prime, > 2. Since x is prime, its only divisors are 1 and itself, since $x > 2$, $x \neq 2$, so $2 \nmid x$ so x is odd. ✓

Back to functions

If $f: A \to B$ is injective then $\exists g: B \to A$ such that $\forall x$,

$$g(f(x)) = x$$

← left inverse of f.



Proof: to prove something exists easiest way is to construct it (tell me what it is)

$$\text{Let } g(y) = \begin{cases} \text{if } y = f(x) & \Rightarrow x \\ \text{else, choose } x_y \in A & \Rightarrow x_y \end{cases}$$

Need to show: ① $\boxed{g \text{ is a function}}$? ← need that there's only one x.

② for all x $g(f(x)) = x$.

Choose arb. x. Then true by def^n

QED

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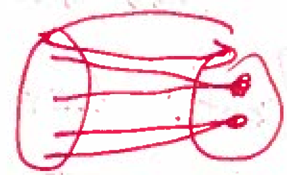
Right inverse?

If $f: A \to B$ is surjective then $\exists g: B \to A$ such that $\forall y \in B$, $f(g(y)) = y$



<u>Proof:</u> let $g(y)$ be defined as follows.

Since f is onto, $\forall y$, $\exists x_y$ s.t. $f(x_y) = y$. Define $g(y) = x_y$. Then $f(g(y)) = f(x_y) = y$ by defn.

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<u>Claim:</u> if f has a left-inverse, then f is injective.

<u>Proof:</u> Let g be the left-inverse, so we know $g(f(x)) = x$.

Wts. f is injective, i.e. $\forall x, y$, if $f(x) = f(y)$ then $x = y$.

well, since $f(x) = f(y)$ $\underbrace{g(f(x))}_{x} = \underbrace{g(f(y))}_{y}$.

so $x = y$ as required. QED

Claim: if f has right inverse then f is surjective.

Claim: if $f: A \to B$ is injection, $g: A \to B$ is surjective then $\exists\, h: A \to B$ which is bijective.
