Inclusion-Exclusion

CS 2800: Discrete Structures, Fall 2014

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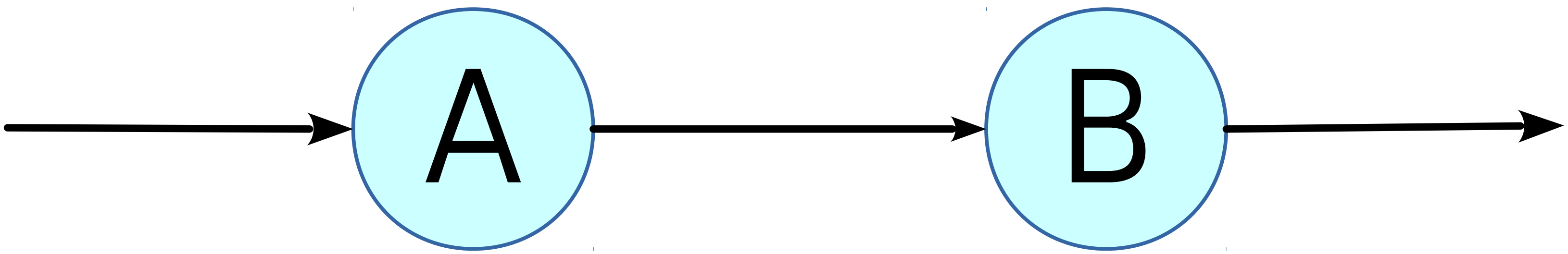
telegraph.co.uk


ONE DOES NOT SIMPLY
HIT THE SNOOZE BUTTON ONCE
quickmeme.com

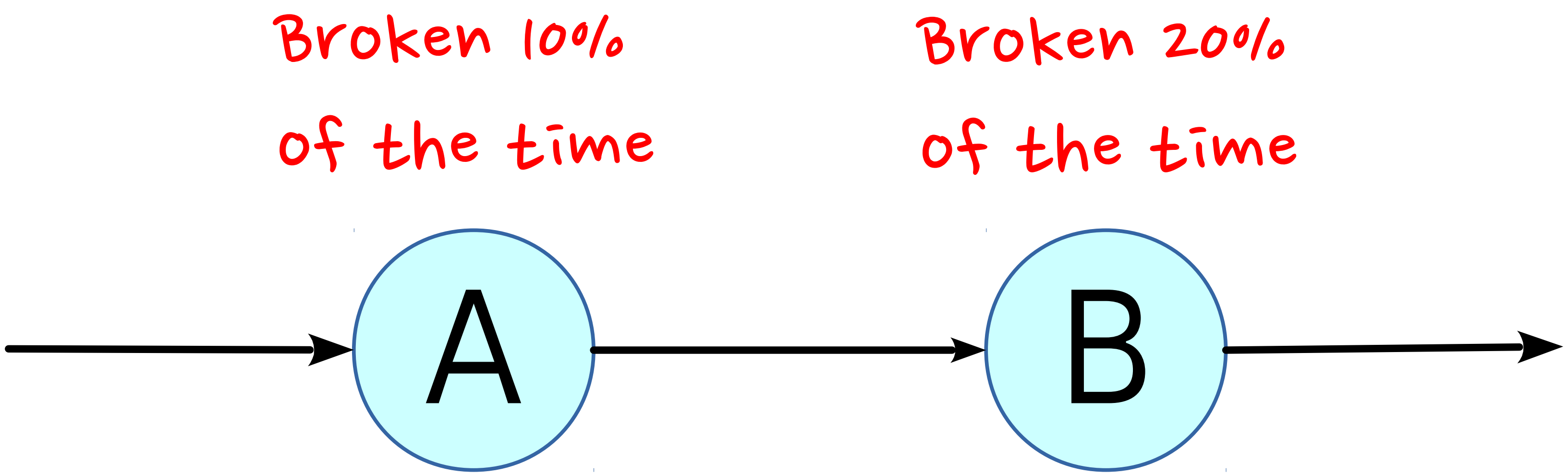
Probability of general celebration in class

- Buses are delayed/cancelled **10%** of the time during the winter
- I oversleep **20%** of the time

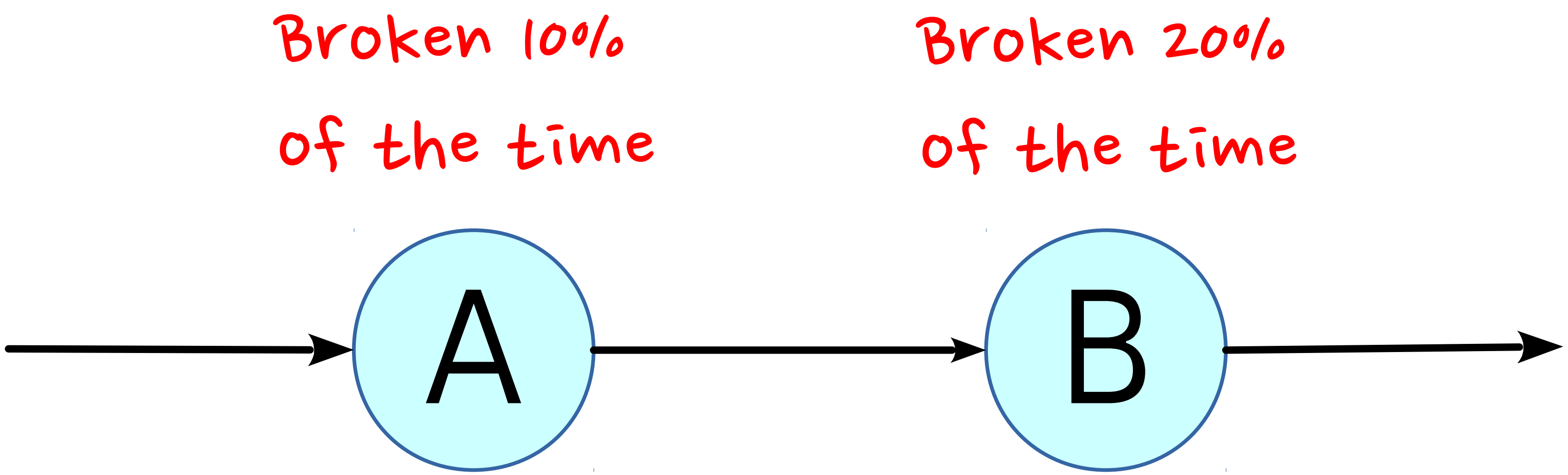
Analogous: a serial circuit



Analogous: a serial circuit



What's the probability no current is flowing?



Modeling

- A: event that buses are delayed
  - (or first component breaks)
- B: event that I oversleep
  - (or second component breaks)
- $Late = A \cup B$: event that I am late
  - (or current is blocked)

Probability of a Union

- Kolmogorov's 3rd Axiom guarantees a simple formula for the probability of the union of mutually exclusive events in a probability space

$$\mathrm{P}(E_1 \cup E_2 \cup E_3 \cup \ldots) = \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) + \ldots$$

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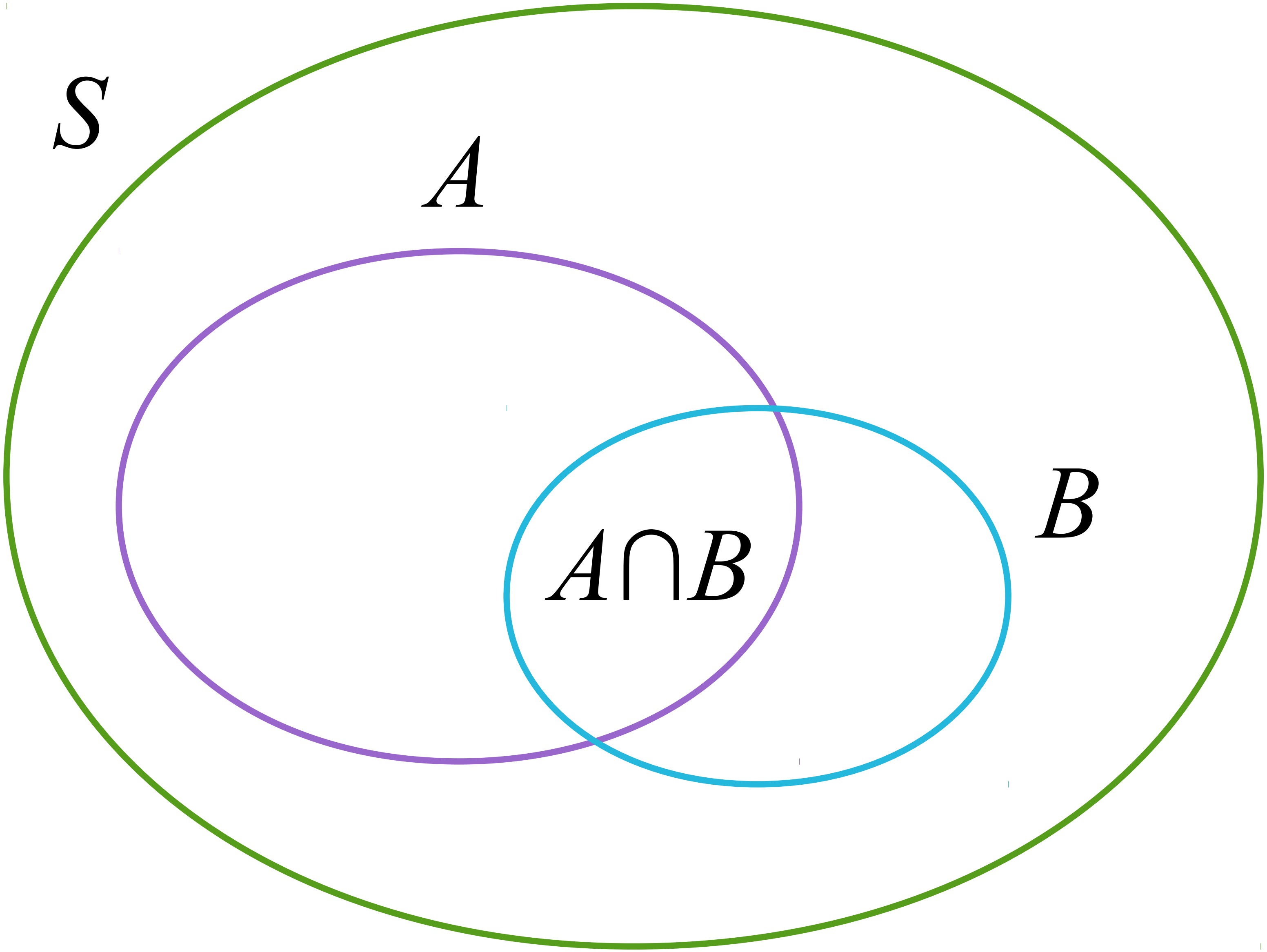
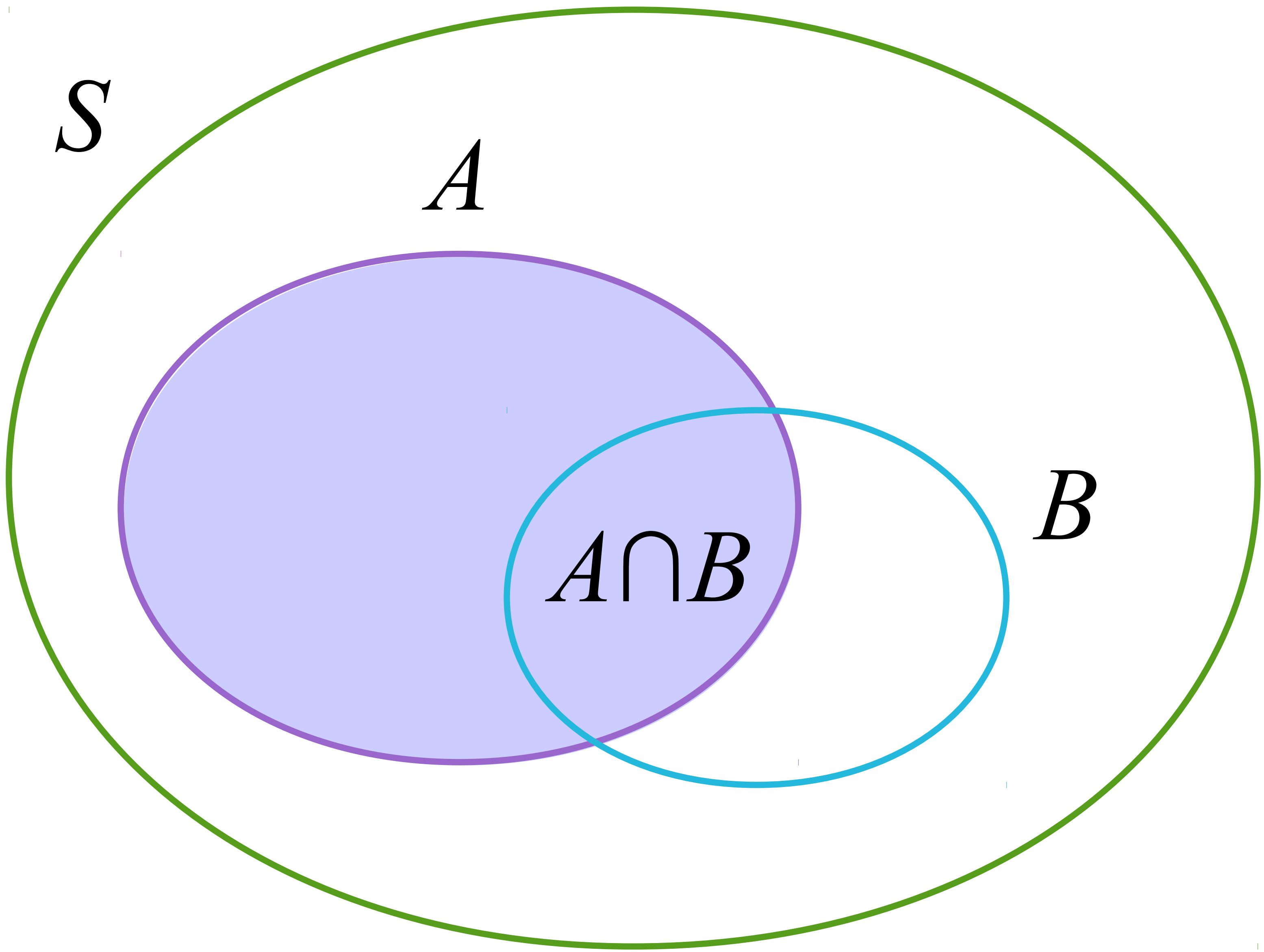
- But what if the events are *not* mutually exclusive?


S
A
B
A∩B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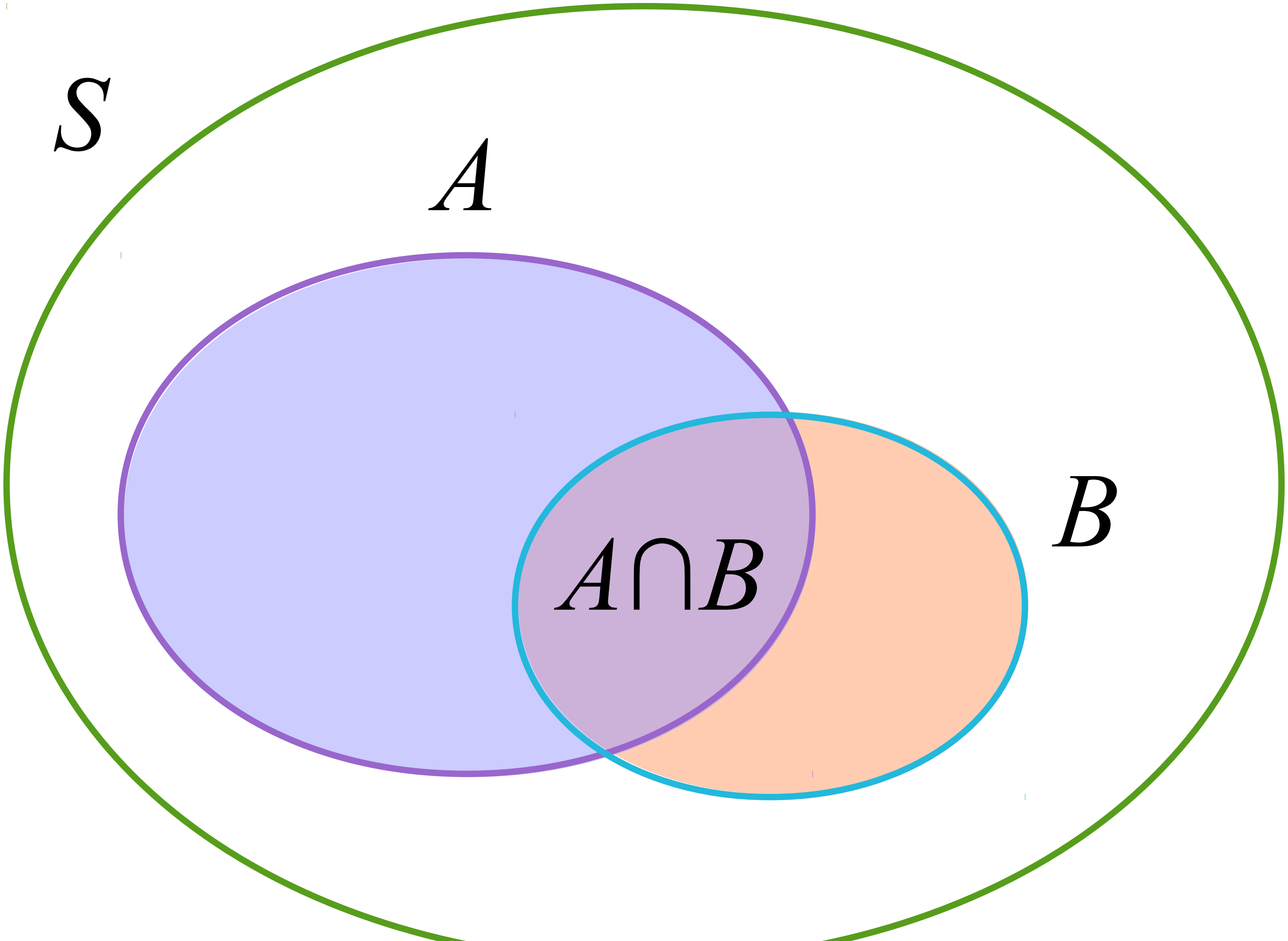
S
A
$A \cap B$
B

Counting Elements

$|A \cup B|$

Counting Elements

$$|A \cup B| = |A \cup (B \setminus A)|$$

Counting Elements

$$\begin{aligned} |A \cup B| &= |A \cup (B \setminus A)| \\ &= |A| + |B \setminus A| \end{aligned}$$

Counting Elements

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\begin{aligned}
|A \cup B| &= |A \cup (B \setminus A)| \\
&= |A| + |B \setminus A| \\
&= |A| + |B \setminus A| + |A \cap B| - |A \cap B|
\end{aligned}
$$

Counting Elements

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\end{aligned}
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Counting Elements

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&= |A| + |(B \setminus A) \cup (A \cap B)| - |A \cap B|
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&= |A| + |B| - |A \cap B|
\end{aligned}
$$

A similar result holds for probabilities

The Inclusion-Exclusion Principle

(for two events)

For two events A, B in a probability space:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

The Inclusion-Exclusion Principle

(for two events)

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$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Don't use this to “prove” Kolmogorov's Axioms!!!

The Inclusion-Exclusion Principle

(for two events)

Proof:

$$P(A \cup B) = P(A \cup (B \setminus A))$$ (set theory)

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$$= P(A) + P(B \setminus A) \qquad \text{(mut. excl., so Axiom 3)}$$

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$$= P(A) + P(B \setminus A) + P(A \cap B) - P(A \cap B)$$

(Adding $0 = P(A \cap B) - P(A \cap B)$)

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(for two events)

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$$= P(A) + P((B \setminus A) \cup (A \cap B)) - P(A \cap B)$$

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(mut. excl., so Axiom 3)

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Will I be late for class?

- $P(Late) = P(A \cup B)$

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- $\mathrm{P}(\textcolor{red}{Late}) = \mathrm{P}(A \cup B)$

  $= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Will I be late for class?

- $P(\textit{Late}) = P(A \cup B)$

  $= P(A) + P(B) - P(A \cap B)$

  $= 0.1 + 0.2 - ???$

Will I be late for class?

- $$\begin{aligned} P(\textit{Late}) &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= 0.1 + 0.2 - ??? \end{aligned}$$

Let's make the modeling assumption
A and B are independent: $P(A \cap B) = P(A)\, P(B)$

Will I be late for class?

- $\text{P}(\color{red}{Late}) = \text{P}(A \cup B)$

$$= \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$

$$= \text{P}(A) + \text{P}(B) - \text{P}(A)\,\text{P}(B)$$

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$$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\,\mathrm{P}(B)$$

$$= 0.1 + 0.2 - 0.1 \times 0.2$$

Will I be late for class?

- $$\begin{aligned} \mathrm{P}(\mathit{Late}) &= \mathrm{P}(A \cup B) \\ &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \\ &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\,\mathrm{P}(B) \\ &= 0.1 + 0.2 - 0.1 \times 0.2 \\ &= \mathbf{0.28} \end{aligned}$$

The Inclusion-Exclusion Principle

(for three events)

For three events A, B, C in a probability space:

$$
\begin{aligned}
\mathrm{P}(A \cup B \cup C) \;=\; & \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) \\
& - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) \\
& + \mathrm{P}(A \cap B \cap C)
\end{aligned}
$$

The Inclusion-Exclusion Principle

For events $A_1, A_2, A_3, \ldots A_n$ in a probability space:

$$P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P\left(A_i\right) - \sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right)$$
$$+ \sum_{1 \le i < j < k \le n} P\left(A_i \cap A_j \cap A_k\right) - \ldots + \left(-1\right)^{n-1} P\left(\cap_{i=1}^{n} A_i\right)$$

The Inclusion-Exclusion Principle

For events $A_1, A_2, A_3, \ldots A_n$ in a probability space:

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$$+ \sum_{1 \le i < j < k \le n} P\left(A_i \cap A_j \cap A_k\right) - \ldots + (-1)^{n-1} P\left(\cap_{i=1}^{n} A_i\right)$$

$$= \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{1,2,\ldots n\} \\ |I| = k}} P\left(\cap_{i \in I} A_i\right) \right)$$