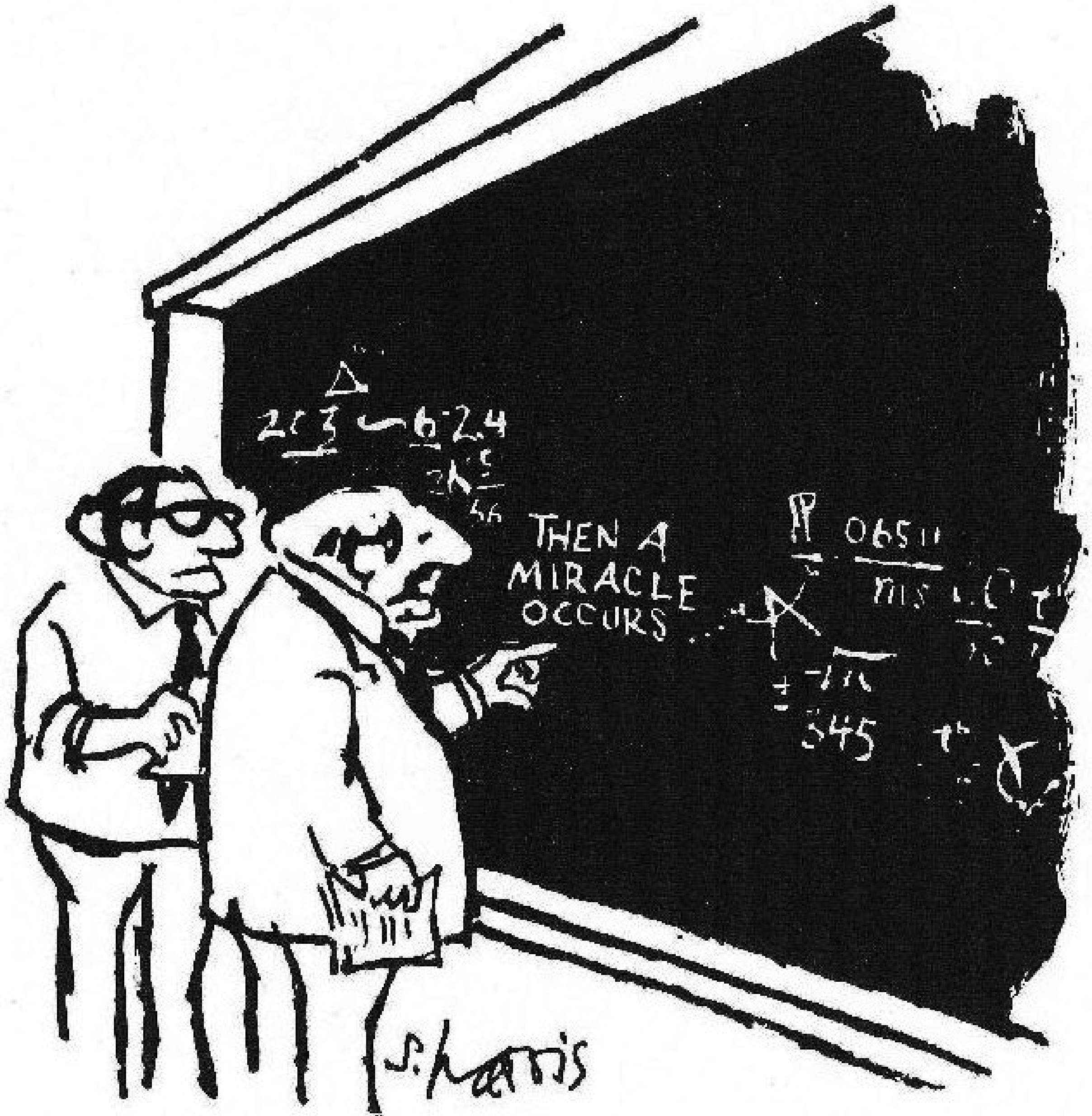
Probability 101

CS 2800: Discrete Structures, Fall 2014

Sid Chaudhuri

But first…



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Sidney Harris

Euclid's Proof of Infinitude of Primes

- Suppose there is a finite number of primes
- Then there is a largest prime, p
- Consider $n = (1 \times 2 \times 3 \times \ldots \times p) + 1$
- n cannot be prime (p is the largest)
- Therefore it has a (prime) divisor $< n$
- But no number from 2 to p divides n
- So n has a prime divisor greater than p

Contradiction!!!

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why?

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Thought for the Day #1

Every positive integer ≥ 2 has at least one prime divisor. How would you prove this?

Thought for the Day #1

Every positive integer ≥ 2 has at least one prime divisor. How would you prove this?

(Equivalently: every non-prime (“composite”) number ≥ 2 has a smaller prime divisor)

Back to Probability...

New York Zoological Society

Thought for the Day #2

After how many years will a monkey produce the Complete Works of Shakespeare with more than 50% probability?

Thought for the Day #2

After how many years will a monkey produce the Complete Works of Shakespeare with more than 50% probability?

(or just an intelligible tweet?)

Elements of Probability Theory

- Outcome
- Sample Space
- Event
- Probability Space


What's the most you ever lost on a coin toss?
Miramax

Heads

Tails

Outcomes

Heads

Tails

Sample Space

Heads

Tails

Sample Space

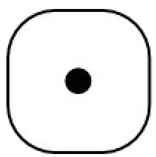
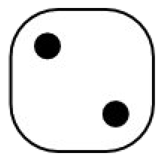
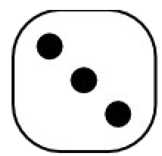
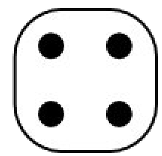
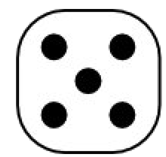
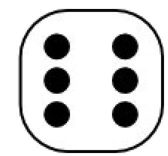
Set of all possible outcomes of an experiment

Some Sample Spaces

- Coin toss: { 

, 

}

- Die roll: {  ,  ,  ,  ,  ,  }

- Weather: {  ,  ,  ,  }

Sample Space

Set of all **<u>mutually exclusive</u>** possible outcomes of an experiment

Event

Subset of sample space

Set Theory

- Set S: unordered collection of elements

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- E. g. $S = \{$ a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z $\}$

  $V = \{$ a, e, i, o, u $\}$

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The set of — all x's — **such that**

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such that

x* is an element of *S

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The set of — all x's — such that — x is an element of S — **and**

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  or $V = \{ x \in S \mid x$ is a vowel $\}$
- V is a subset of S, or $V \subseteq S$

Event

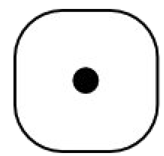
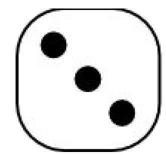
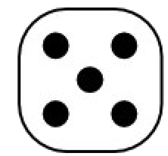
Subset of sample space

Some Events

- Event of a coin landing heads: {  }

Some Events

- Event of a coin landing heads: {  }

- Event of an odd die roll: {  ,  ,  }

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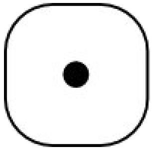
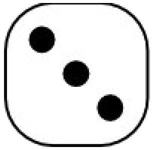
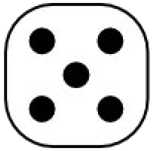
- Event of an odd die roll: {  ,  ,  }
- Event of weather like Ithaca:

{  ,  ,  ,  }

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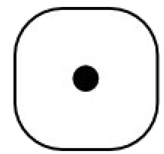
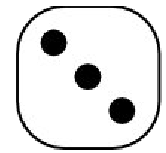
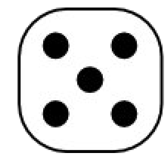
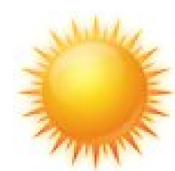
- Event of an odd die roll: {  ,  ,  }

- Event of weather like Ithaca:

  {  ,  ,  ,  }

- Event of weather like California: {  }

Careful!

- The sample space is a **set** (of outcomes)
- An outcome is an **element** of a sample space
- An event is a **set** (a *subset* of the sample space)
  - It can be **empty** (the null event { } or $\varnothing$, which never happens)
  - It can contain a **single** outcome (simple/elementary event)
  - It can be the **entire** sample space (certain to happen)
- Strictly speaking, an outcome is not an event (it's not even an elementary event)

Building New Events from Old Ones

- $A \cup B$ (read 'A union B') consists of all the outcomes in A or in B (or both!)
- $A \cap B$ (read 'A intersection B') consists of all the outcomes in both A and B
- $A \setminus B$ (read 'A minus B') consists of all the outcomes in A but not in B
- A' (read 'A complement') consists of all outcomes not in A (that is, $S \setminus A$)

Probability Space

Sample space S, plus function P assigning real-valued probabilities $P(E)$ to events $E \subseteq S$, satisfying Kolmogorov's axioms

Kolmogorov's Axioms

1. For any event E, we have $P(E) \geq 0$

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3. If events $E_1, E_2, E_3, \ldots$ are pairwise disjoint ("mutually exclusive"), then

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \ldots$$

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Thought for the Day #3

Can you prove, from the axioms, that $P(E) \leq 1$ for all events E?

Equiprobable Probability Space

- All outcomes equally likely (fair coin, fair die...)
- Laplace's definition of probability (only in equiprobable space!)

$$P(E) = \frac{|E|}{|S|}$$

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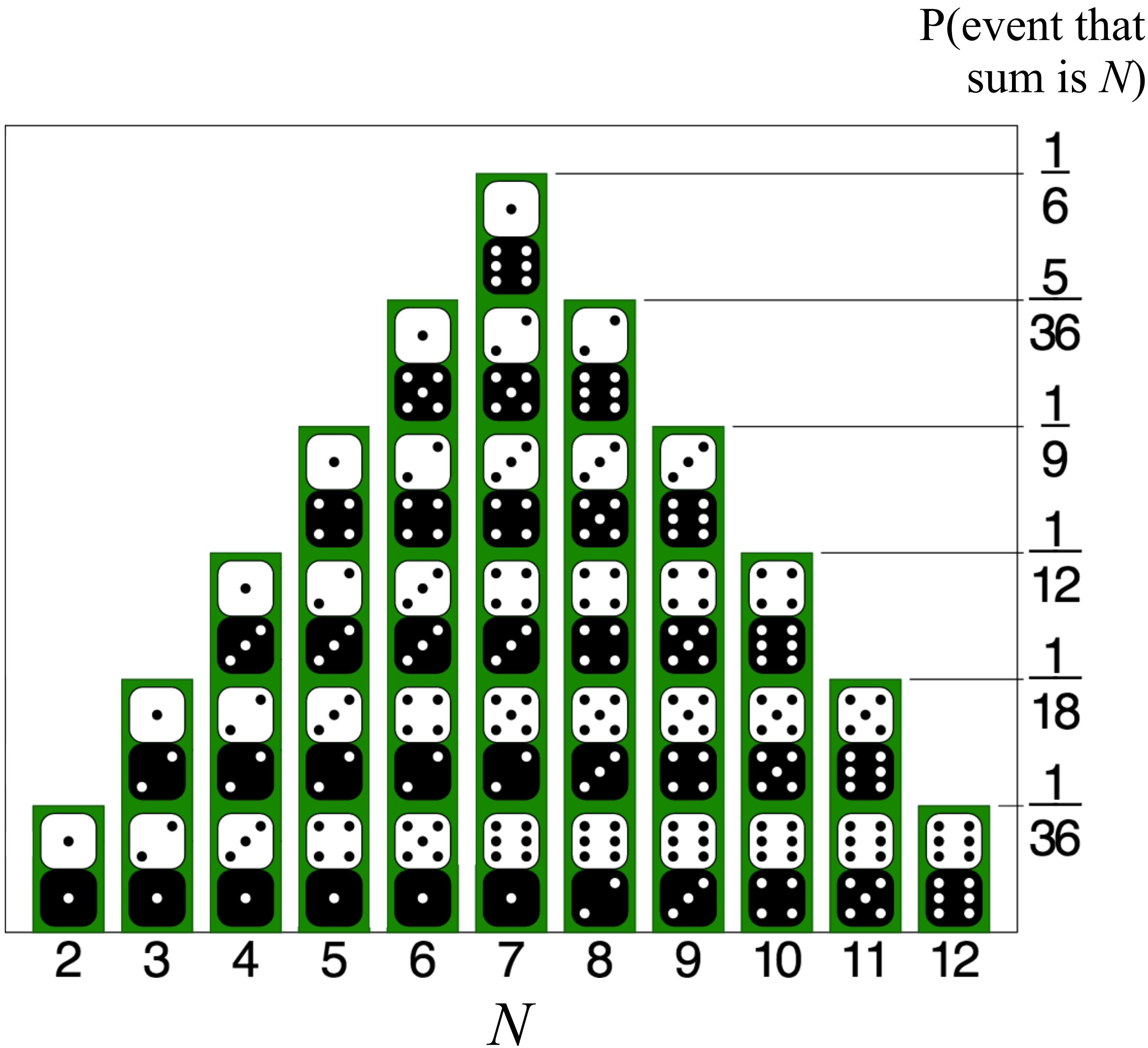
Number of elements (outcomes) in E

Number of elements (outcomes) in S



Tim Stellmach, Wikipedia

Gerolamo Cardano
(1501-1576)

Liar, gambler, lecher, heretic

saderpo@glogster