Discrete Structures CS2800 Fall 2014	Homework 7 Induction and Regular Expressions

1. The city of Lineapolis is shaped like a circle, with a wall along the perimeter. All roads in the city are perfectly straight lines, extending from wall to wall. Every two roads intersect at a single point within the city, and no three roads (or two roads and the wall) intersect at the same point. Every maximal region within the city, not intersected by any road or wall, constitutes a city block.

   When Lineapolis was built, the city planners faced the problem of deciding how many blocks there would be if they built n roads. You will help them solve this problem.

   (a) Draw the first three cases $n = 1$, $n = 2$, $n = 3$. For each n, is the number of city blocks the same regardless of how you draw the roads? If not, show a counterexample. If yes, write down the number of blocks for $n = 1, 2$ and 3.

   (b) Propose a relationship between n, the number of roads, and $B(n)$, the number of blocks. No justification is necessary at this stage. (**Hint:** Study the structure of the pictures you drew in part (a) and draw more pictures if you want.)

   (c) Prove the relationship you proposed in (b).

   (d) Lineapolis has two competing lines of coffeeshops. Each of the two companies lobbies for exclusive rights to open shops in each city block. The city council, however, wants to ensure that citizens always have a choice of morning brew. As a compromise, the council votes that every block will be allocated to a single coffee company, but no two adjacent blocks (that is, blocks sharing a non-zero length of road) will be allocated to the same company. Prove that this arrangement can always be satisfied, regardless of the number of roads in Lineapolis.

2. In this question we will give very precise inductive definitions of some familiar string operations. Recall that the set of strings Σ^* is defined inductively by the following rules:

   - $\epsilon \in \Sigma^*$
   - for any $x \in \Sigma^*$ and $a \in \Sigma$, $xa \in \Sigma^*$.

   (a) Give an inductive definition of the length function $\ell : \Sigma^* \to \mathbb{N}$.

   (b) Give an inductive definition of the concatenation function $c : \Sigma^* \times \Sigma^* \to \Sigma^*$, which takes two strings x and y and returns their concatenation xy.

   (c) Prove inductively that $\ell(c(x, y)) = \ell(x) + \ell(y)$

   (d) Prove inductively that $c(c(x, y), z) = c(x, c(y, z))$

   (e) Prove inductively that $\hat{\delta}(\hat{\delta}(q, x), y) = \hat{\delta}(q, c(x, y))$

3. Are the following sets regular? If so, give a regular expression that matches them. If not, prove that the set is not regular.

   (a) $\{x \in \{0, 1\}^* \mid |x| \geq 5\}$

   (b) Σ^*

   (c) Strings that alternate between 0 and 1

   (d) Strings with the same number of 0s and 1s

   (e) Strings of balanced parentheses (e.g. "()()", "((()()))", and "", but not "())(()").

4. Prove or disprove: if L is a regular set and $L' \subseteq L$, then L' is also regular.