Discrete Structures	Homework 5
CS2800 Fall 2014	Cardinality

1. For each of the following pairs of sets, state whether $|A| < |B|$, $|A| = |B|$, or $|A| > |B|$. Justify your answer, for example by giving a bijection (no need to prove that it is a bijection), or by referring to examples from class. As usual, $\mathbb{N}$ is the set of all natural numbers $\{1, 2, 3, \dots\}$ and $\mathbb{Z}$ is the set of all integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
   (a) $A = \{1, 2, 3\}$, $B = \{a, b, c\}$.
   (b) $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$.
   (c) $A = \{1, 2, 3\}$, $B = \mathbb{Z}$.
   (d) $A = \mathbb{N}$, $B = \mathbb{Z}$.
   (e) $A = \mathbb{N}$, $B = \{0, 1\}^*$, the set of all finite strings of 0's and 1's.
   (f) $A = \mathbb{N}$, $B = 2^{\mathbb{N}}$ (the power set of $\mathbb{N}$)

2. In class, we gave three related definitions:

   Definition of $|\cdot| = |\cdot|$: We say that $|A| = |B|$ if there exists a bijection from A to B.

   Definition of $|\cdot| \leq |\cdot|$: We say that $|A| \leq |B|$ if there exists an injection from A to B.

   Definition of $|\cdot| \geq |\cdot|$: We say that $|A| \geq |B|$ if there exists a surjection from A to B.

   This is very suggestive notation; for example you might expect that if $|A| \leq |B|$ then $|B| \geq |A|$. In this question you will prove that these definitions make sense together.
   (a) Prove that $|A| \leq |B|$ if and only if $|B| \geq |A|$.
   (b) Prove that $|A| = |A|$.
   (c) Prove that if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$.
   (d) Prove that $|A| = |B|$ if and only if both $|A| \leq |B|$ and $|A| \geq |B|$.

3. Given functions f and g from $\mathbb{R}^+$ to $\mathbb{R}^+$ let $F(f, g)$ be the following statement:

   $F(f, g) =$"there is some $c > 0$ such that for all $x > 1$, $f(x) \leq cg(x)$."

   Let $f_1 : x \mapsto x^2$ and let $f_2 : x \mapsto x$. Two students are arguing about whether $F(f_1, f_2)$. Student one offers the following proof that $F(f_1, f_2)$:

   "I wish to prove that there exists $c > 0$ such that for all $x > 1$, $f_1(x) \leq cf_2(x)$. Let $c = x$. Since $x > 1$, we know $c > 0$. Moreover, for any $x > 1$, we have

   $$cf_2(x) = x \times x = x^2 \geq x^2 = f_1(x)$$

   Thus $F(f_1, f_2)$."

   Student two offers the following rebuttal, claiming that $\neg F(f_1, f_2)$.

   "I wish to prove that $\neg F(f_1, f_2)$. In other words, I need to show that for all $c > 0$, there exists an $x > 1$ such that $f_1(x) > cf_2(x)$. Choose an arbitrary $c > 0$. Let $x = c + 1$. Then

   $$f_1(x) = f_1(c + 1) = c^2 + 2c + 1 > c^2 + c = cf_2(c + 1) = cf_2(x)$$

   Note also that since $c > 0$, $x > 1$. Thus there exists such an $x > 1$ for each $c > 0$, so I have shown $\neg F(f_1, f_2)$."

   The student who is incorrect has proved something different than what they are claiming. Which student is it, and what have they proved?