1. As we have seen, our intuition about probability is often wrong. One famous example is the "Monty Hall" game. The game is played as follows: there are three doors. One of the doors is randomly selected (each with equal probability), and a valuable prize (say \$100) is placed behind the door. The other two doors typically conceal goats, which are assumed to be worthless for this question.

   A contestant then chooses one of the doors at random (with equal probability). Before the door is opened, the host opens one of the other doors to reveal a goat (again, if the host has a choice of different goat doors to open, you can assume (s)he chooses one at random, with equal probability). The contestant is then given the opportunity to switch. Some people argue that switching makes no difference; clearly the goat is behind one of the two doors, so you have a 50% chance of selecting the goat if you switch. Others argue that the probability that the contestant chooses the right door initially is only 1/3, so there should be a 2/3 probability that the prize is behind the other door.

   In this problem you'll use the formal tools of probability to determine which answer is correct.

   (optional) If you haven't seen this problem before, guess which argument is correct.

   (a) Write down the set of outcomes in a sample space that models this game.

   (b) Give the probability of each of the simple events in this space. Do not make any assumptions about the behavior of the contestant.

   (c) Write down the event "a player who always switches wins". Compute the probability of this event using your definition of P from part (b) and Kolmogorov's axioms.

   (d) Likewise, compute the probability of the event "a player who never switches wins".

   (e) Suppose the contestant has to pay \$50 to play. What are the expected net winnings of the player who always switches?

   (f) Suppose we modify the game so that the contestant has to pay an additional fee to switch. How much should we charge to give the two strategies the same expected value?

2. Each of the following relations has domain $\{x, y, z\}$ and codomain $\{1, 2, 3\}$. For each, state whether it is a function or not. If it is a function, state (i) its image, (ii) whether it is surjective or not, (iii) whether it is injective or not. If it is not a function, state why not.

   (a) $\{(x, 1), (y, 2), (x, 3), (z, 2)\}$

   (b) $\{(x, 1), (y, 2)\}$

   (c) $\{(x, 3), (y, 2), (z, 1)\}$

   (d) $\{(x, 1), (y, 1), (z, 1)\}$

3. Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is injective (one-to-one) or not. If it is not injective, give a counterexample. Note that $\mathbb{Z}$ is the set of all integers: negative, positive or zero.

   (a) $f(n) = n - 1$

   (b) $f(n) = n^2 + 1$

   (c) $f(n) = n^3$

   (d) $f(n) = \lceil n/2 \rceil$ ($\lceil x \rceil$ denotes the smallest integer greater than or equal to x)

4. For this problem, if X and Y are sets, we will write $[X \to Y]$ to denote the set of all functions with domain X and codomain Y.

   (a) Give a bijection from $[X \to [Y \to Z]]$ to $[X \times Y \to Z]$.

   (b) Give a bijection from the power set of X to $[X \to \{\text{Mike}, \text{Sid}\}]$.

   (c) Assuming X is isomorphic to Y, give a bijection from $[X \to Z]$ to $[Y \to Z]$.